



The range of the double fibration transform

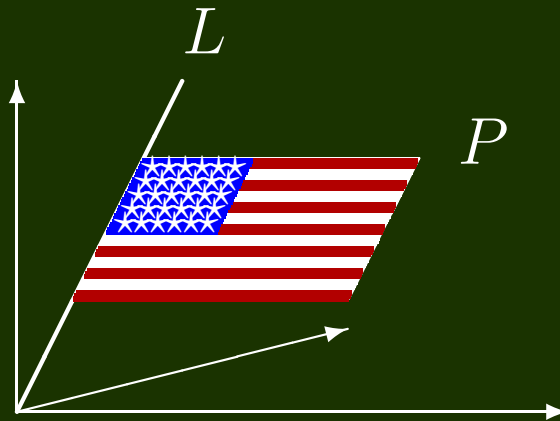
Michael Eastwood

[joint work with Joseph Wolf]

Australian National University

Flag manifolds

- $\mathbb{C}\mathbb{P}_n = \{L \subset \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} L = 1\}$
- $\text{Gr}_k(\mathbb{C}^{n+1}) = \{L \subset \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} L = k\}$
- $F_{1,2}(\mathbb{C}^3) = \{L \subset P \subset \mathbb{C}^3 \mid \dim_{\mathbb{C}} L = 1, \dim_{\mathbb{C}} P = 2\}$

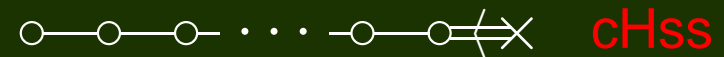


$$\mathbb{C}\mathbb{P}_2 = \text{SL}(3, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \quad F_{1,2}(\mathbb{C}^3) = \text{SL}(3, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

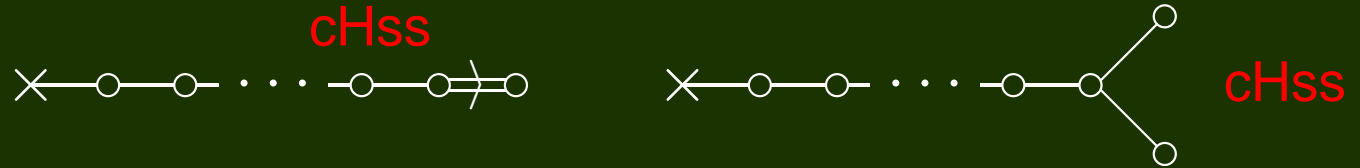
Flag manifolds cont'd (generalised)

$$\text{SL}(n+1, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix} \right\} = \text{Dynkin diagram with } n \text{ nodes}$$

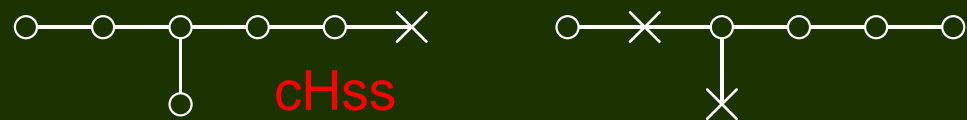
Langrangian Grassmannians



Quadrics



Exceptional examples



Flag domains

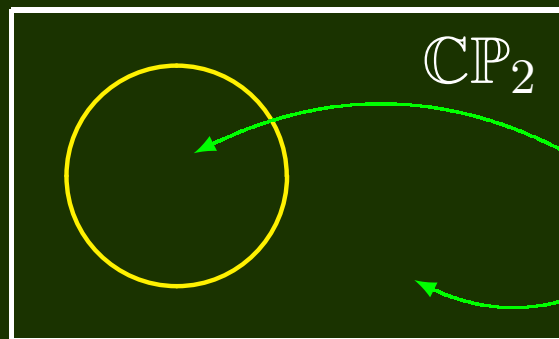
Flag manifold $Z = G/Q$ $\begin{cases} G = \text{complex simple Lie group} \\ Q = \text{complex parabolic subgroup} \end{cases}$

New ingredient $G_0 = \text{real form of } G$. Let it act on Z .

Theorem (Wolf) There are finitely many orbits.

Corollary There are open orbits. \leftarrow Flag domain $D \subseteq Z$

Example $Z = \mathbb{C}P_2 = \text{SL}(3, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$ $G_0 = \text{SU}(2, 1)$



two possible flag domains $D \subset \mathbb{C}P_2$

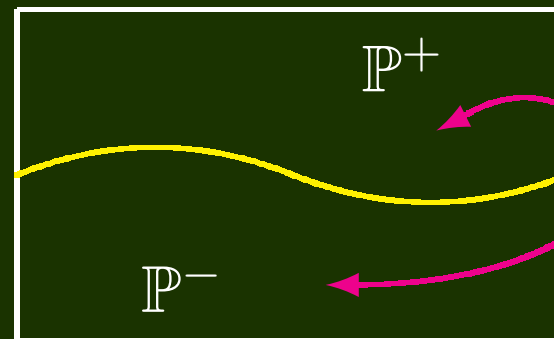
Twistor space

$$\mathbb{CP}_3 = \mathrm{SL}(4, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \right\}$$

$$G_0 = \mathrm{SU}(4) = \mathrm{Spin}(6) \Rightarrow D = \mathbb{CP}_3 \quad \underline{\text{one orbit}}$$

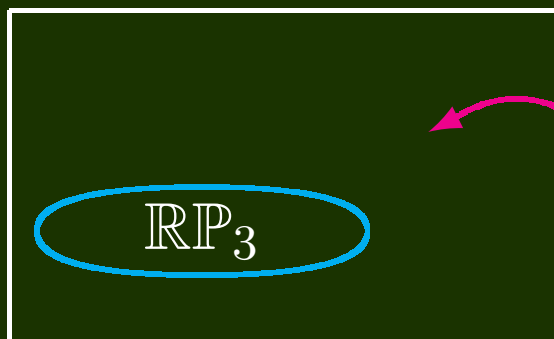
$$G_0 = \mathrm{SL}(2, \mathbb{H}) = \mathrm{Spin}(5, 1) \Rightarrow D = \mathbb{CP}_3 \quad \underline{\text{one orbit!!}}$$

$$G_0 = \mathrm{SU}(2, 2) = \mathrm{Spin}(4, 2)$$



two open orbits

$$G_0 = \mathrm{SL}(4, \mathbb{R}) = \mathrm{Spin}(3, 3)$$

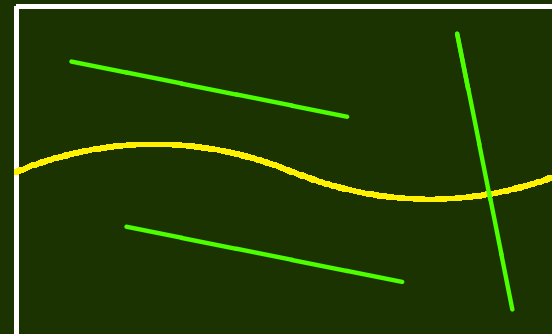


one open orbit

Compactified Minkowski space

$$\mathbb{M} \equiv \text{Gr}_2(\mathbb{C}^4) = \text{SL}(4, \mathbb{C}) / \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

$$G_0 = \text{SU}(2, 2) = \text{Spin}(4, 2)$$



three
open
orbits

Flag domains: \mathbb{M}^{++} \mathbb{M}^{+-} \mathbb{M}^{--}
(six orbits in total, three open and one closed)

Closed orbit = compactified Minkowski space
 $\text{SU}(2, 2)$ acts by conformal automorphisms

$\mathbb{M} =$ complexified compactified Minkowski space

Cycle spaces

Ingredients so far


- flag manifold $Z = G/Q$ $\times \text{---} \circ \text{---} \times \text{---} \circ \text{---} \circ \text{---} \dots$
- flag domain $D \subseteq Z$ for $G_0 \subset G$

Let $K_0 =$ a maximal compact subgroup of G_0 .

EG $S(U(2) \times U(2)) \subset SU(2, 2)$

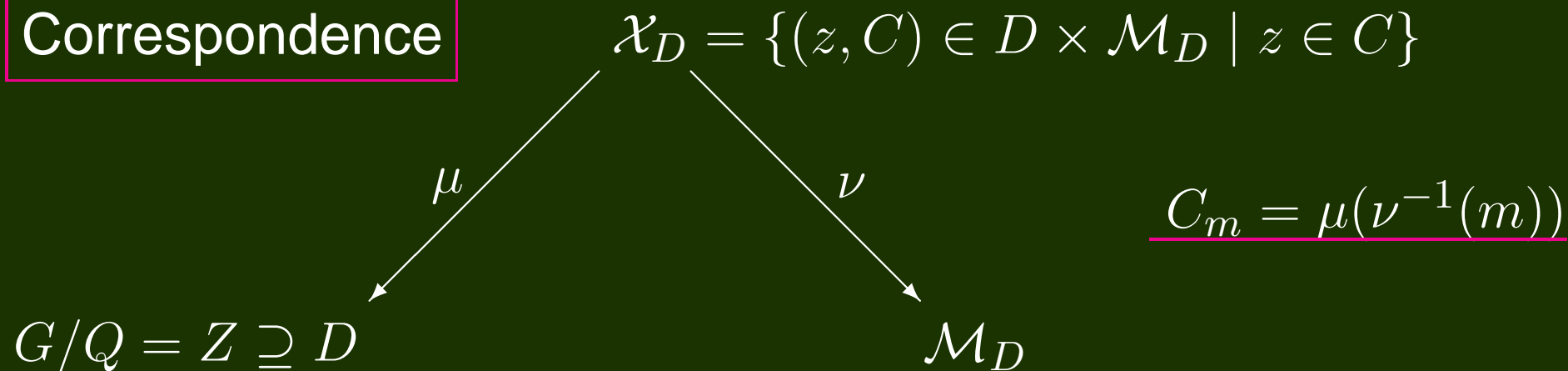
Theorem (Wolf) There is exactly one K_0 -orbit $C_0 \subset D$ that is a complex submanifold. \leftarrow base cycle

Definition $\mathcal{M}_D \equiv \{gC_0 \subset D \mid g \in G\}^\circ \leftarrow$ cycle space

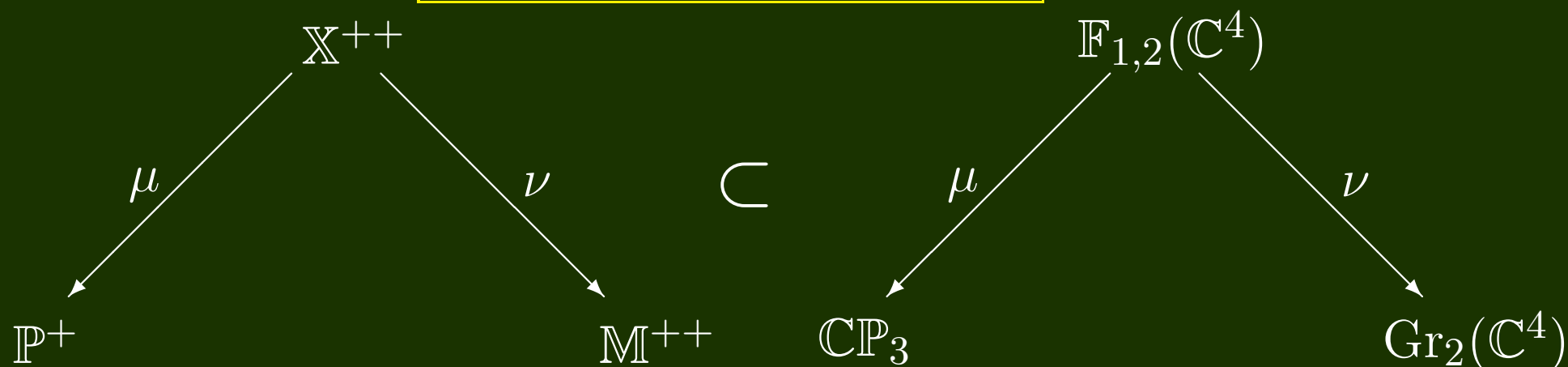
EG $D = \mathbb{P}^+$  $\implies \mathcal{M}_D = \mathbb{M}^{++}$
(maybe misleading)

Double fibrations

Correspondence



Twistor Correspondence



Types of flag domains/cycle spaces

$$\mathcal{M}_Z \equiv \{gC_0 \mid g \in G\} = G/J, \quad \text{where } J \equiv \{g \in G \mid gC_0 = C_0\}$$

NB $J \supseteq K$ (often equal) $\therefore \dim_{\mathbb{C}} \mathcal{M}_Z \leq \dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} K_0$

$$\mathcal{M}_D^{\text{open}} \subset \mathcal{M}_Z \rightsquigarrow \mathcal{M}_D \text{ is } \underline{\text{holomorphic}}$$

\mathcal{M}_D might not be G_0 -homogeneous!!

Perhaps: $D = Z$ (G_0 compact or rare cases).

Otherwise: trichotomy via decomposition of K -modules

$$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+ \text{ and } \mathfrak{j} = \mathfrak{k} \oplus \mathfrak{s}_+ \quad \text{Eg: } \text{SU}(2, 2) \text{ on } \mathbb{P}^+$$

$$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+ \text{ and } \mathfrak{j} = \mathfrak{k} \quad \text{Eg: } \text{SU}(2, 2) \text{ on } \mathbb{M}^{+-}$$

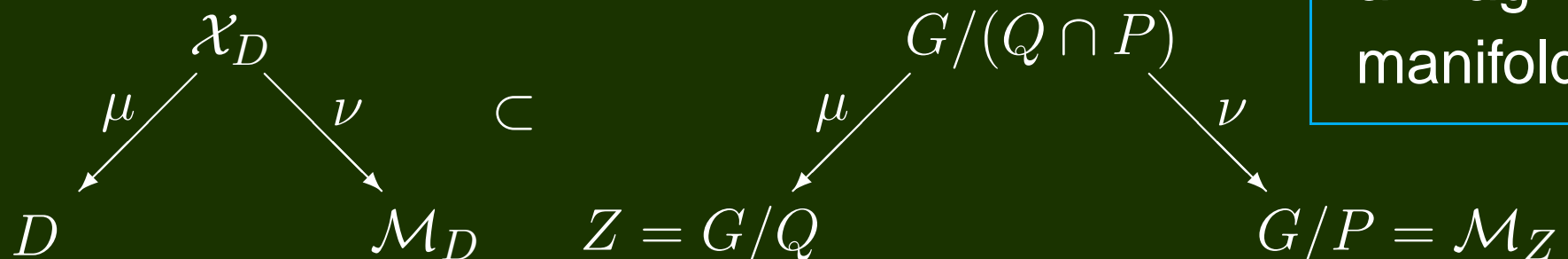
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \text{ and } \mathfrak{j} = \mathfrak{k} \quad \text{Eg: } \text{SL}(4, \mathbb{R}) \text{ on } \mathbb{CP}_3 \setminus \mathbb{RP}_3$$

Hermitian holomorphic case

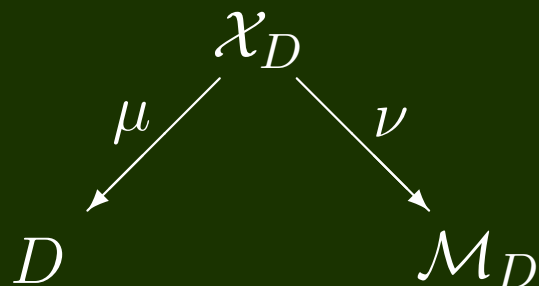
$$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+ \quad \text{and} \quad \mathfrak{j} = \mathfrak{k} \oplus \mathfrak{s}_+$$

Then

- $\mathfrak{s}_- = (\mathfrak{s}_+)^*$ Abelian subalgebras of \mathfrak{g}
- \mathfrak{j} is parabolic, $J \equiv P$ and $\mathcal{M}_Z = G/P$ is a flag manifold
- \mathcal{M}_Z is a compact Hermitian symmetric space
- $\mathcal{M}_D \subset \mathcal{M}_Z = G/P$ is G_0 -homogeneous, i.e. a flag domain
- $\mathcal{M}_D = G_0/K_0$ is a bounded symmetric domain
- $\dim_{\mathbb{C}} \mathcal{M}_D = (\dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} K_0)/2$



Double fibration transform: generalities



Theorem (... , Fels-Huckleberry-Wolf)

- \mathcal{M}_D is Stein (and contractible)
- the fibres of μ are contractible

Let $s = \dim_{\mathbb{C}} C_0 = \dim_{\mathbb{C}}(\text{fibres of } \nu)$.

Let $\mathbb{E} \rightarrow Z \supset D$ be a homogeneous vector bundle on $Z = G/Q$.

Then $m \mapsto H^s(C_m, \mathcal{O}(\mathbb{E}|_{C_m}))$ is a hvb on $G/J = \mathcal{M}_Z \supset \mathcal{M}_D$.

Call it \mathbb{E}' ($\mathcal{O}(\mathbb{E}') \equiv \nu_*^s(\mu^* \mathcal{O}(\mathbb{E}))$)

Double fibration transform $\mathcal{P} : H^s(D, \mathcal{O}(\mathbb{E})) \rightarrow \Gamma(\mathcal{M}_D, \mathcal{O}(\mathbb{E}'))$

$H^s(D, \mathcal{O}(\mathbb{E})) \ni \omega \mapsto \omega|_{C_m} \in H^s(C_m, \mathcal{O}(\mathbb{E}|_{C_m}))$ for $m \in \mathcal{M}_D$

cf. Radon transform or X-ray transform

Double fibration transform: specifics

Example $G = \mathrm{SL}(4, \mathbb{C})$ $Z = \mathbb{CP}_3$ $G_0 = \mathrm{SU}(2, 2)$

$D = \mathbb{P}^+$ $\mathcal{M}_D = \mathbb{M}^{++}$ $\mathbb{E} = \Omega^3$ [i.e. $\mathcal{O}(\mathbb{E}) = \mathcal{O}(-4)$]

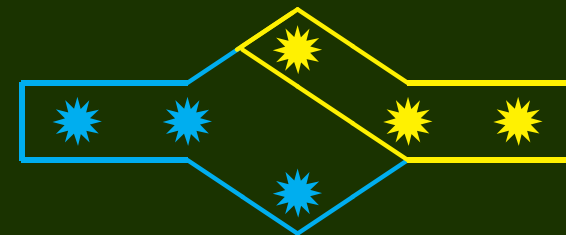
$$\mathbb{E} = \overset{-4}{\times} \overset{0}{\circ} \overset{0}{\circ} \implies \mu^* \mathbb{E} = \overset{-4}{\times} \overset{0}{\times} \overset{0}{\circ} \implies \mathbb{E}' = \nu_*^1 \mu^* \mathbb{E} = \overset{2}{\circ} \overset{-3}{\times} \overset{0}{\circ} = \Omega_+^2$$

Cohomological machinery \rightsquigarrow

$\mathcal{P} : H^1(\mathbb{P}^+, \Omega^3) \rightarrow \Gamma(\mathbb{M}^{++}, \Omega_+^2)$ satisfies

- \mathcal{P} is injective
- range(\mathcal{P}) = $\{\phi \in \Gamma(\mathbb{M}^{++}, \Omega_+^2) \text{ s.t. } d\phi = 0\}$ (Maxwell).

$$\Omega^0 \xrightarrow{d} \Omega^1 \begin{cases} \nearrow \Omega_+^2 \\ \searrow \Omega_-^2 \end{cases} \begin{cases} \nearrow \Omega^3 \\ \searrow \Omega^3 \end{cases} \rightarrow \Omega^4$$



Double fibration transform: injectivity

Theorem (Huckleberry & Wolf) \mathbb{E} 'sufficiently negative'

$$\Rightarrow \mathcal{P} : H^s(D, \mathcal{O}(\mathbb{E})) \rightarrow \Gamma(\mathcal{M}_D, \mathcal{O}(\mathbb{E}')) \text{ is injective}$$

Theorem (E & Wolf) In the Hermitian holomorphic case,
 \mathbb{E} 'sufficiently negative'

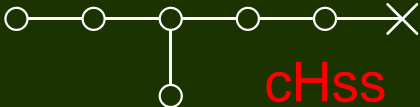
$$\Leftrightarrow \Omega_Z^{\text{top}} \otimes \mathbb{E}^* \text{ is non-negative}$$

$$\Omega_Z^{\text{top}} \otimes \mathbb{E}^* = \begin{array}{cccccc} a & b & c & d & e & \dots \\ \times & \circ & \times & \circ & \circ & \dots \end{array} \quad a, c, \dots \geq 0$$

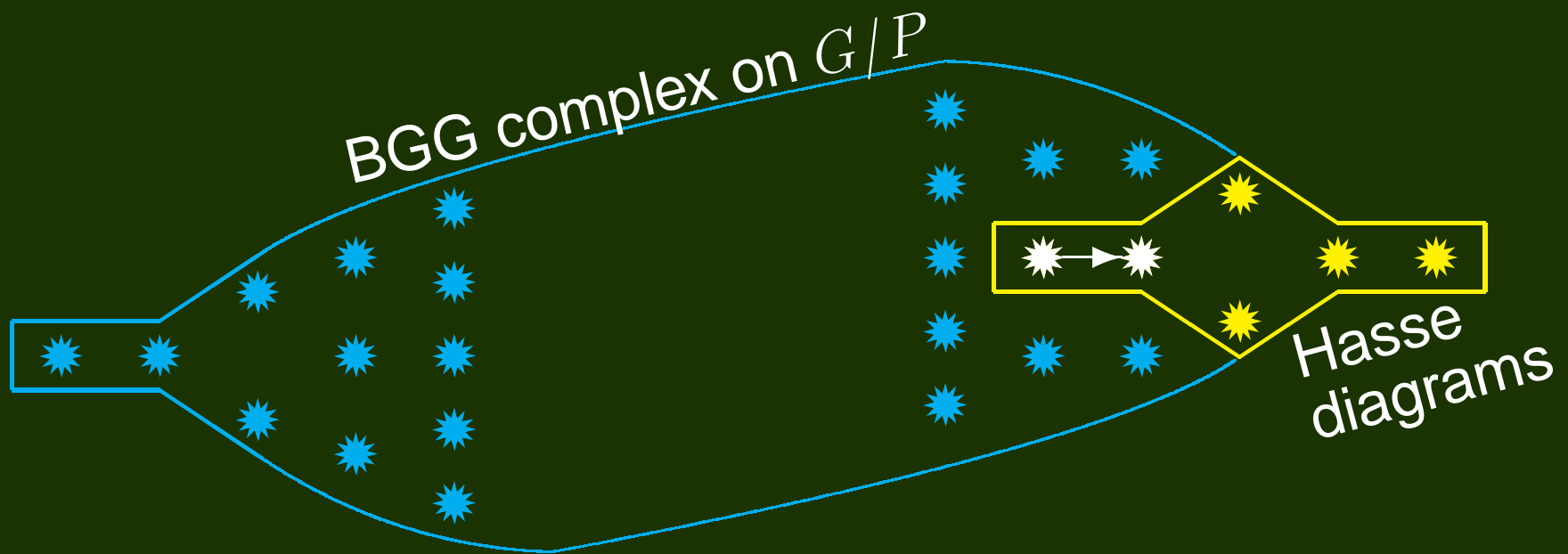
$\mathbb{E} \rightsquigarrow \Omega_Z^{\text{top}} \otimes \mathbb{E}^*$ OK via diagrams (effective algorithm).

Double fibration transform: range

Theorem (E & Wolf) In the Hermitian holomorphic case,

$$\mathcal{M}_D \subset \mathcal{M}_Z = G/P =$$


$$(\text{Fibres of } \mu) \subset Q/(P \cap Q) =$$



THANK YOU