The Penrose Transform for Complex Projective Space

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Twistor space

Linear algebra: $J : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies

- $J^2 = -\mathrm{Id}$
- $J \in \mathrm{SO}(4)$

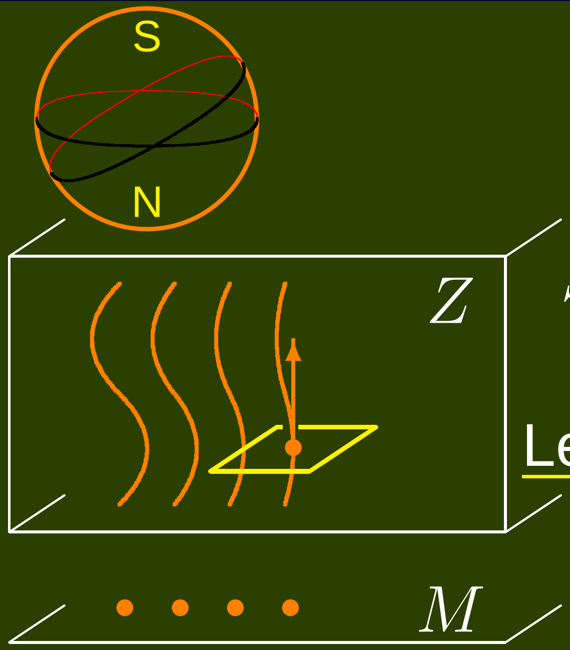
$$\iff J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & 0 & -u \\ w & -v & u & 0 \end{bmatrix}$$

$u^2 + v^2 + w^2 = 1$, 2-sphere! $= \mathrm{SO}(4)/\mathrm{U}(2)$

Twistor space

$$\begin{array}{ccccl} & & \lightning & & \\ S^2 & \to & Z & = & \text{bundle of almost Hermitian structures} \\ & & \tau\downarrow & & \\ & & M & = & \text{oriented Riemannian } 4\text{-manifold} \end{array}$$

Twistor space cont'd



$S^2 = \mathbb{CP}_1$

Levi-Civita

$\Longrightarrow$ Z is almost-complex

Theorem Z is complex iff M is anti-self-dual, i.e. $W_+ = 0$ where Riem $= W_+ + W_- +$ Ric

Example $M = S^4$ with round metric $R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$

$$\begin{array}{ccc} Z & = & \mathbb{CP}_3 \\ \tau\downarrow & & \downarrow \\ S^4 & = & \mathbb{HP}_1 \end{array} \qquad \text{cf. Hopf} \qquad \begin{array}{ccc} S^3 & \xrightarrow{2:1} & \mathbb{RP}_3 \\ \downarrow & & \downarrow \\ S^2 & = & \mathbb{CP}_1 \end{array}$$

Example: $\mathbb{CP}_2$ with Fubini-Study metric

Curvature $R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$

Orientation decree: J_{ab} should be anti-self-dual

$$\begin{array}{ccc} Z = \mathbb{F}_{1,2}(\mathbb{C}^3) & \ni & (L, P) \\ \tau \downarrow & & \downarrow \\ \mathbb{CP}_2 & \ni & L^\perp \cap P \end{array}$$

SU(3)-invariant

NB fibres of τ are holomorphic so can define $\Lambda^{1,0}_\mu$ by

$$0 \to \Lambda^{1,0}_\mu \to \Lambda^{0,1}_Z \to \Lambda^{0,1}_\tau \to 0.$$

$\Lambda^{1,0}_\mu$ has rank 2 and is holomorphic along the fibres of τ

$$\Lambda^{1,0}_\mu \to \Lambda^{0,1}_Z \xrightarrow{\bar\partial} \Lambda^{0,2}_Z \to \Lambda^{0,1}_\tau \otimes \Lambda^{1,0}_\mu$$

The Penrose transform

For any twistor space $\tau : Z \to M$, a spectral sequence

$$E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}) \Longrightarrow H^{p+q}(Z, \mathcal{O})$$

where $\Lambda_\mu^{p,0} \equiv \bigwedge^p \Lambda_\mu^{1,0}$ (holomorphic along the fibres of τ).

Computation $\Longrightarrow$

$$\tau_*^q \Lambda_\mu^{p,0} = \begin{array}{l} q\uparrow \\ \begin{array}{ccccc} 0 & & 0 & & 0 \\ | & & & & \\ \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d_+} & \Lambda^2_+ \end{array} \longrightarrow p \end{array}$$

Therefore,

$$H^r(Z, \mathcal{O}) \cong H^r\Big[0 \to \underset{r=0}{\Gamma(M, \Lambda^0)} \to \underset{r=1}{\Gamma(M, \Lambda^1)} \to \underset{r=2}{\Gamma(M, \Lambda^2_+)} \to 0\Big]$$

Consequences

$\boxed{M = S^4}$ $\quad H^r(\mathbb{CP}_3, \mathcal{O}) = H^r(\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}) = \begin{cases} \mathbb{C} & \text{for } r = 0, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$\boxed{0 \to \mathbb{C} \to \Gamma(S^4, \Lambda^0) \xrightarrow{d} \Gamma(S^4, \Lambda^1) \xrightarrow{d_+} \Gamma(S^4, \Lambda^2_+) \to 0}$$

$\boxed{M = \mathbb{CP}_2}$ $\quad H^r(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}) = H^r(\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{0}{\times}) = \begin{cases} \mathbb{C} & \text{for } r = 0, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$\boxed{0 \to \mathbb{C} \to \Gamma(\mathbb{CP}_2, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{d_+} \Gamma(\mathbb{CP}_2, \Lambda^2_+) \to 0}$$

NB $\;\Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{d_-} \Gamma(\mathbb{CP}_2, \Lambda^2_-) \ni J_{ab}$ $\;$ not surjective!

The Penrose transform cont'd

For V a holomorphic vector bundle on Z, a spectral sequence

$$E_1^{p,q} = \Gamma(M, \tau_*^q \Lambda_\mu^{p,0}(V)) \Longrightarrow H^{p+q}(Z, \mathcal{O}(V))$$

where $\Lambda_\mu^{p,0}(V) \equiv \Lambda_\mu^{p,0} \otimes V$ (holomorphic along the fibres of τ).

EG $V = \kappa \rightsquigarrow$

$$\tau_*^q \Lambda_\mu^{p,0}(\kappa) = \begin{array}{l} q\uparrow \\ \Lambda_+^2 \xrightarrow{\quad d \quad} \Lambda^3 \xrightarrow{\quad d \quad} \Lambda^4 \\ 0 \;\;\text{———}\;\; 0 \;\;\text{———}\;\; 0 \longrightarrow p \end{array}$$

Therefore,

$$H^r(Z, \mathcal{O}(\kappa)) \cong H^r\big[0 \to \underset{r=1}{\Gamma(M, \Lambda_+^2)} \to \underset{r=2}{\Gamma(M, \Lambda^3)} \to \underset{r=3}{\Gamma(M, \Lambda^4)} \to 0\big]$$

Consequences

$\boxed{M = S^4}$ $\quad H^r(\mathbb{CP}_3, \mathcal{O}(\kappa)) = H^r(\overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}) = \begin{cases} \mathbb{C} & \text{for } r = 3, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$\boxed{0 \to \Gamma(S^4, \Lambda^2_+) \xrightarrow{d} \Gamma(S^4, \Lambda^3) \xrightarrow{d} \Gamma(S^4, \Lambda^4) \xrightarrow{\int} \mathbb{C} \to 0}$$

$\boxed{M = \mathbb{CP}_2}$ $\quad H^r(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}) = H^r(\overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{-2}{\times}) = \begin{cases} \mathbb{C} & \text{for } r = 3, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$\boxed{0 \to \Gamma(\mathbb{CP}_2, \Lambda^2_+) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^3) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^4) \xrightarrow{\int} \mathbb{C} \to 0}$$

NB $J_{ab} \in \ker : \Gamma(\mathbb{CP}_2, \Lambda^2_-) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^3)$ (generates $H^2(\mathbb{CP}_2, \mathbb{C})$)

Some elliptic complexes on $\mathbb{CP}_2$

R.T. Smith, M. Rumin & N. Seshadri, L.-S. Tseng & S.-T. Yau

$$\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d^{(2)}} & \Lambda^2_\perp & \xrightarrow{d} & \Lambda^3 & \xrightarrow{d} & \Lambda^4 \\
\| & & \| & & \downarrow & & & & & & \\
\Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d_+} & \Lambda^2_+ & & & & & &
\end{array}$$

where $d^{(2)}$ is $\omega_{ab} \mapsto J^{cd}(\nabla_a \nabla_c \omega_{bd} - \nabla_b \nabla_c \omega_{ad})$

Corollary of Penrose transform

$$\boxed{0 \to \mathbb{C} \to \Gamma(\mathbb{CP}_2, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_2, \Lambda^2_\perp)}$$

is exact (but local cohomology of $\Lambda^0 \to \Lambda^1 \to \Lambda^2_\perp$ is $\mathbb{C}$).

Deformations?

What about $H^r(Z,\Theta)$ where $\Theta \equiv \mathcal{O}(TZ)$?

$$\tau_*^q \Lambda_\mu^{p,0}(\Theta) \rightsquigarrow \begin{array}{cccccc} q\uparrow & & & & & \\ 0 & & 0 & & 0 & \\ | & & & & & \\ \Lambda^1 & \xrightarrow{\nabla_\circ} & \odot_\circ^2\Lambda^1 & \xrightarrow{\nabla_{\circ+}^{(2)}} & \mathbb{W}_+ & \xrightarrow{\quad p} \\ \parallel & & \parallel & & \parallel & \\ \square\ \text{rk}=4 & & \square\square_\circ\ \text{rk}=9 & & \boxplus_{\circ+}\ \text{rk}=5 & \end{array}$$

where

$$X_a \mapsto \tfrac{1}{2}\nabla_a X_b + \tfrac{1}{2}\nabla_b X_a - \tfrac{1}{4}\nabla_c X^c g_{ab}$$

$$\parallel$$

$$\pi_{\square\square_\circ}\left[\nabla_a X_b\right]$$

$$h_{ab} \mapsto \pi_{\boxplus_{\circ+}}\left[\nabla_a \nabla_b h_{cd}\right]$$

if M is Einstein

Therefore,

$$H^r(Z,\Theta) \cong H^r\Big[0 \to \underset{r=0}{\Gamma(M,\square)} \to \underset{r=1}{\Gamma(M,\square\square_\circ)} \to \underset{r=2}{\Gamma(M,\boxplus_{\circ+})} \to 0\Big]$$

Consequences

$\boxed{M = S^4}$ $\quad H^r(\mathbb{CP}_3, \Theta) = H^r(\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}) = \begin{cases} \mathfrak{sl}(4,\mathbb{C}) & \text{for } r = 0, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$0 \to \mathfrak{sl}(4,\mathbb{C}) \to \Gamma(S^4, \square) \xrightarrow{\nabla_\circ} \Gamma(S^4, \square\!\square_\circ) \xrightarrow{\nabla^{(2)}_{\circ+}} \Gamma(S^4, \boxplus_{\circ+}) \to 0$$

$\boxed{M = \mathbb{CP}_2}$

$H^r(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) = H^r\left(\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{1}{\times} + \begin{matrix} \overset{2}{\times}\!\!-\!\!\!-\!\!\overset{-1}{\times} \\ \oplus \\ \overset{-1}{\times}\!\!-\!\!\!-\!\!\overset{2}{\times} \end{matrix}\right) = \begin{cases} \mathfrak{sl}(3,\mathbb{C}) & \text{for } r = 0, \\ 0 & \text{else} \end{cases}$

$\Longrightarrow$ an exact sequence

$$0 \to \mathfrak{sl}(3,\mathbb{C}) \to \Gamma(\mathbb{CP}_2, \square) \xrightarrow{\nabla_\circ} \Gamma(\mathbb{CP}_2, \square\!\square_\circ) \xrightarrow{\nabla^{(2)}_{\circ+}} \Gamma(\mathbb{CP}_2, \boxplus_{\circ+}) \to 0$$

More elliptic complexes on $\mathbb{CP}_2$

$$\begin{array}{ccccccccccc}
\square & \xrightarrow{\nabla} & \square\!\square & \xrightarrow{\nabla_\perp^{(2)}} & \boxplus_\perp & \xrightarrow{\nabla_\perp^{(2)}} & \boxplus_\perp & \xrightarrow{\nabla^{(2)}} & \square\!\square & \xrightarrow{\nabla} & \square \\
\| & & \downarrow & & \downarrow & & & & & & \\
\square & \xrightarrow{\nabla_\circ} & \square\!\square_\circ & \xrightarrow{\nabla_{\circ+}^{(2)}} & \boxplus_{\circ+} & & & & & &
\end{array}$$

where $\nabla_\perp^{(2)}$ is $h_{ab} \mapsto \pi_{\boxplus_\perp}\left[\nabla_a \nabla_b h_{cd} + g_{ab} h_{cd}\right]$

Corollary of Penrose transform

$\left[\text{since } \nabla_\perp(\lambda g_{ab}) = \frac{1}{2}\lambda(g_{ac}g_{bd} - g_{bc}g_{ad}) + \frac{1}{5}\lambda(J_{ac}J_{bd} + \cdots)\right]$

$$0 \to \mathfrak{sl}(3,\mathbb{C}) \to \Gamma(\mathbb{CP}_2, \square) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_2, \square\!\square) \xrightarrow{\nabla_\perp^{(2)}} \Gamma(\mathbb{CP}_2, \boxplus_\perp)$$

is exact ($\rightsquigarrow$ infinitesimal Blaschke rigidity (cf. Tsukamoto))

Higher projective spaces?

Based on joint work with Joseph Wolf, there are choices.

- As a fibration $\mathbb{F}_{1,2}(\mathbb{C}^4) \ni (L, P) \mapsto L^\perp \cap P \in \mathbb{CP}_3$
- As a correspondence defined by incidence

$$\mathbb{F}_{1,3}(\mathbb{C}^4) \times \mathbb{CP}_3 \ni \big((L,H),\ell\big) \text{ s.t. } \ell \in H \text{ and } L \in \ell^\perp$$

Penrose transforms $H^r(Z, \mathcal{O}) \rightsquigarrow$ elliptic complexes on $\mathbb{CP}_3$

$$\Lambda^0 \to \Lambda^1 \to \begin{array}{c}\Lambda^{0,2}\\ \oplus \\ \Lambda^{1,1}_\perp\end{array} \to \Lambda^{1,2}_\perp$$

$$\Lambda^0 \to \Lambda^1 \to \Lambda^2 \to \begin{array}{c}\Lambda^{1,2}\\ \oplus \\ \Lambda^{2,1}\end{array} \to \Lambda^{2,2}_\perp$$

respectively.

THANK YOU

HAPPY BIRTHDAY SIGURDUR!