A canonical connection on sub-Riemannian contact manifolds

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Contact geometry

contact distribution

$$\begin{array}{ccc} H & \hookrightarrow & TM \\ \operatorname{rank} = 2n & & \operatorname{rank} = 2n+1 \end{array}$$

$$0 \to H \to TM \to L^* \to 0 \quad (\leftrightsquigarrow \; 0 \to L \to \Lambda^1 \to \Lambda^1_H \to 0)$$

de Rham

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \Lambda^1_H & & \Lambda^2_H & & \Lambda^3_H & & \\ & & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \\ \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \longrightarrow & \Lambda^2 & \longrightarrow & \Lambda^3 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & L & & \Lambda^1_H \otimes L & & \Lambda^2_H \otimes L & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Levi form

$\mathcal{L} \in \operatorname{Hom}(L, \Lambda^2_H)$

non-degenerate

Sub-Riemannian contact geometry

contact distribution $H \hookrightarrow TM$

sub-Riemannian metric g_{ab} on H

contact form $\theta \in \Gamma(L) \hookrightarrow \Gamma(\Lambda^1)$

$$0 \to L \to \Lambda^1 \to \Lambda^1_H \to 0$$

$$\Gamma(H) = \{X \in \Gamma(TM) \text{ s.t. } X \lrcorner \theta = 0\}$$

$\rightsquigarrow$ symplectic form $J_{ab} \equiv \mathcal{L}_{ab}(\theta) \in \Gamma(\Lambda^2_H)$ on H

$g_{ab} \rightsquigarrow \theta$ e.g. normalise $J^{ab} J_{ab} = 2n$

Reeb field $T \lrcorner \theta = 1$, $T \lrcorner d\theta = 0$ $\rightsquigarrow$ splitting $0 \to H \to TM \underset{\curvearrowleft}{\to} L^* \to 0$

$\rightsquigarrow$ Riemannian metric on M ! $\rightsquigarrow \ldots$ ☠

Cauchy-Riemann contact geometry

CR geometry (of hypersurface type)

contact distribution $H \hookrightarrow TM$

sub-almost complex structure $J_a{}^b$ on H $\quad J_a{}^b J_b{}^c = -\delta_a{}^c$

suppose partially integrable $\quad g_{ab} \equiv J_a{}^c J_{bc}$ is symmetric

NB then g_{ab} is non-degenerate with inverse g^{ab} and

$$g_{ab} = J_a{}^c J_{bc} \quad J_a{}^b = J_{ac} g^{bc} \quad J_{ab} = J_a{}^c g_{bc} \quad J_{ac} J^{bc} = \delta_a{}^b$$

M is strictly pseudo-convex $\iff$ g_{ab} is positive definite

NB automatic when $n = 1$

Three-dimensional contact geometry

CR-geometry + contact form = pseudo-Hermitian geometry

sub-Riemannian contact geometry

agree when $\dim M = 3$!!!

generally, there's an inclusion

Three-dimensional sub-Riemannian contact geometry

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \longrightarrow \Lambda^1_H, \qquad d_H : \Lambda^0 \to \Lambda^1_H$$

partial connection

$\nabla_H : V \to \Lambda^1_H \otimes V$ s.t. $\nabla_H(f\sigma) = f\nabla_H\sigma + d_H f \otimes \sigma$

A canonical partial connection

partial torsion

$$\begin{array}{ccccc} \Lambda^1 & \xrightarrow{\nabla_H} & \Lambda^1_H \otimes \Lambda^1 & \rightarrow & \Lambda^1_H \otimes \Lambda^1_H \\ \| & & & & \downarrow \\ \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \longrightarrow & \Lambda^2_H \end{array}$$

commutes?

- Contact form θ $\rightsquigarrow$ Reeb field T $\Big\} \rightsquigarrow \Lambda^1 = \Lambda^1_H \oplus L = \Lambda^1_H \oplus \Lambda^0$

- $$\Lambda^1 = \begin{array}{ccc} \Lambda^1_H & \xrightarrow{\nabla_H} & \Lambda^1_H \otimes \Lambda^1_H \\ \oplus & \nearrow_{\mathcal{L}} & \oplus \\ \Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H \end{array} = \Lambda^1_H \otimes \Lambda^1$$

free from partial torsion and preserves the metric

Proof: usual Levi Civita algebra but on H.

Compare with Tanaka-Webster

$$\Lambda^1 = \begin{array}{c}\Lambda^1_H\\ \oplus\\ \Lambda^0\end{array} \begin{array}{c}\xrightarrow{\nabla}\\ \\ \xrightarrow{d}\end{array} \begin{array}{c}\Lambda^1\otimes\Lambda^1_H\\ \oplus\\ \Lambda^1\end{array} = \Lambda^1\otimes\Lambda^1$$

$$\nabla^{\text{T-W}}$$

$$\text{Hom}(L,\Lambda^2_H) \ni \mathcal{L}$$

$$\oplus$$

torsion $\text{Hom}(\Lambda^1,\Lambda^2) = \Lambda^1_H \oplus \Lambda^1_H$

$$\oplus$$

Thus, canonical partial connection $\Lambda^2_H \oplus \bigodot^2\Lambda^1_H$ $\tau^{\text{T-W}}$

$$\boxed{(\nabla^{\text{T-W}} + \mathcal{L})_H}$$

Promote to full connection

$\rightsquigarrow$ new torsion $R^{\text{T-W}}$

"solves the equivalence problem"

THE END

THANK YOU