

# MATHEMATICAL FRAGMENTS

BEING

FACSIMILES OF HIS UNFINISHED PAPERS

RELATING TO THE

## THEORY OF GRAPHS

BY THE LATE

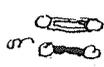
W. K. CLIFFORD

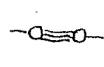
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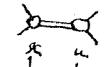
MACMILLAN AND CO.

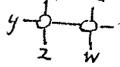
1881

# Theory of a single Quartic Form

 =  $i$  Invariant.

 =  $\frac{1}{2} \delta \cdot C$  This is unaltered when the variables are interchanged;  $\therefore$  divides by  $C$ .

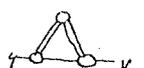
 =  $H$  Hessian; to make it symmetrical we must write  $\frac{x}{y} \frac{u}{v} + \frac{1}{6} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$

 =  $\frac{1}{2} \frac{y}{z} \frac{v}{w} \cdot C_u^x + \frac{1}{2} \frac{x}{z} \frac{u}{w} \cdot C_v^y + \frac{1}{2} \frac{x}{y} \frac{u}{v} \cdot C_w^z$

 =  $j$

 =  $\frac{1}{y} \frac{v}{z} \cdot C_w^x + \frac{1}{z} \frac{u}{w} \cdot C_v^y$

 =  $-\frac{1}{2} \frac{t}{z} \frac{v}{w} + \frac{1}{2} \frac{t}{w} \frac{v}{z} \cdot C_w^z$  |  $\frac{w}{z} \frac{v}{z} + \frac{t}{z} \frac{v}{w} = \frac{t}{z} \frac{v}{w} \cdot C_w^z$

 =  $\frac{1}{2} \frac{t}{z} \frac{v}{w} \cdot C_v^y$

 =  $\frac{1}{2} \cdot C_y^x$  because it is unaltered when  $x, y$  are interchanged. Multiplying by  $(xy)$ .

$\therefore \frac{1}{2} \frac{t}{z} \frac{v}{w} = \frac{t}{z} \frac{v}{w} \cdot C_w^z - \frac{1}{2} \frac{t}{z} \frac{v}{w} \cdot C_w^z$

we find  $-\frac{t}{z} \frac{v}{w} = -2 \frac{t}{z} \frac{v}{w}$ ; but  $\frac{t}{z} \frac{v}{w}$  vanishes identically; therefore

 = 0 identically; and consequently also

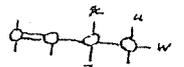
 = 0 identically

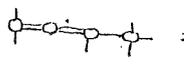
Now the form  $\frac{x}{y} \frac{u}{v} + \frac{1}{6} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$  being symmetrical; we shall get similar results, <sup>with opposite signs</sup> whether we multiply by  $\frac{x}{y} \frac{u}{v}$  or by  $\frac{x}{y} \frac{v}{u}$ . In the first case

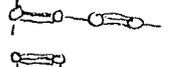
we get  $\frac{x}{y} \frac{u}{v} + \frac{1}{6} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\} = + \frac{1}{6} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$ . In the second we get  $\frac{x}{y} \frac{v}{u} + \frac{1}{6} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$

$\frac{x}{y} \frac{v}{u} = -\frac{1}{2} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$ . Hence  $\frac{x}{y} \frac{v}{u} = -\frac{1}{2} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$ ,  $\frac{x}{y} \frac{u}{v} = -\frac{1}{2} \left\{ C_u^x \cdot C_v^y + C_v^x \cdot C_u^y \right\}$ , etc

The sextic covariant. All the allies of  $f$  with  $T$  are easily seen to be reducible; viz

 =  $\frac{1}{2} \left\{ \frac{x}{z} \frac{u}{w} \cdot C_v^y - \frac{1}{2} \frac{y}{z} \frac{v}{w} \cdot C_u^x + \frac{1}{2} \frac{x}{z} \frac{u}{w} \cdot C_v^y \right\}$

 =  $-\frac{1}{2} \left\{ \frac{x}{z} \frac{u}{w} \cdot C_v^y + \frac{y}{z} \frac{v}{w} \cdot C_u^x \right\}$

 =  $\frac{1}{2} \left\{ \frac{x}{z} \frac{u}{w} \cdot C_v^y + \frac{y}{z} \frac{v}{w} \cdot C_u^x \right\}$

 =  $\frac{1}{2} \left\{ \frac{x}{z} \frac{u}{w} \cdot C_v^y + \frac{y}{z} \frac{v}{w} \cdot C_u^x \right\} \cdot C$

Quartic.

 is quadriinvariant  $i$

 unsymmetrical quartic covariant; to make it symmetrical we must add a multiple of   $\supset C$   
It is then the Hessian.

 say  $(xyzu)(xyzv)$ , being unaltered by interchange of  $u, v$   
is multiple of  $\mu v$  It is at once found to be  $\frac{1}{2}$  

Two or more quadrics.

 joint invariant  $\mathcal{D}_{ab}$

  $-\frac{1}{2} \supset \mathcal{D}_{ab}$  cov $\mathcal{E}$   $\mathcal{F}_{ab}$

all other forms derived from two quadrics are clearly reducible

e.g.  =   $\phi$  =  $\frac{1}{2}$    $\phi$ ;  =   $\mathcal{D}_{ab}$  =  $2$  

 is reducible. For   $-\frac{1}{2} \supset \mathcal{D}_{ab}$  being symmetrical, we get the same result by multiplying  $\phi$  into either end

of it; thus   $-\frac{1}{2} \phi \mathcal{D}_{ab}$  =  +  $\frac{1}{2} \phi \mathcal{D}_{ab}$

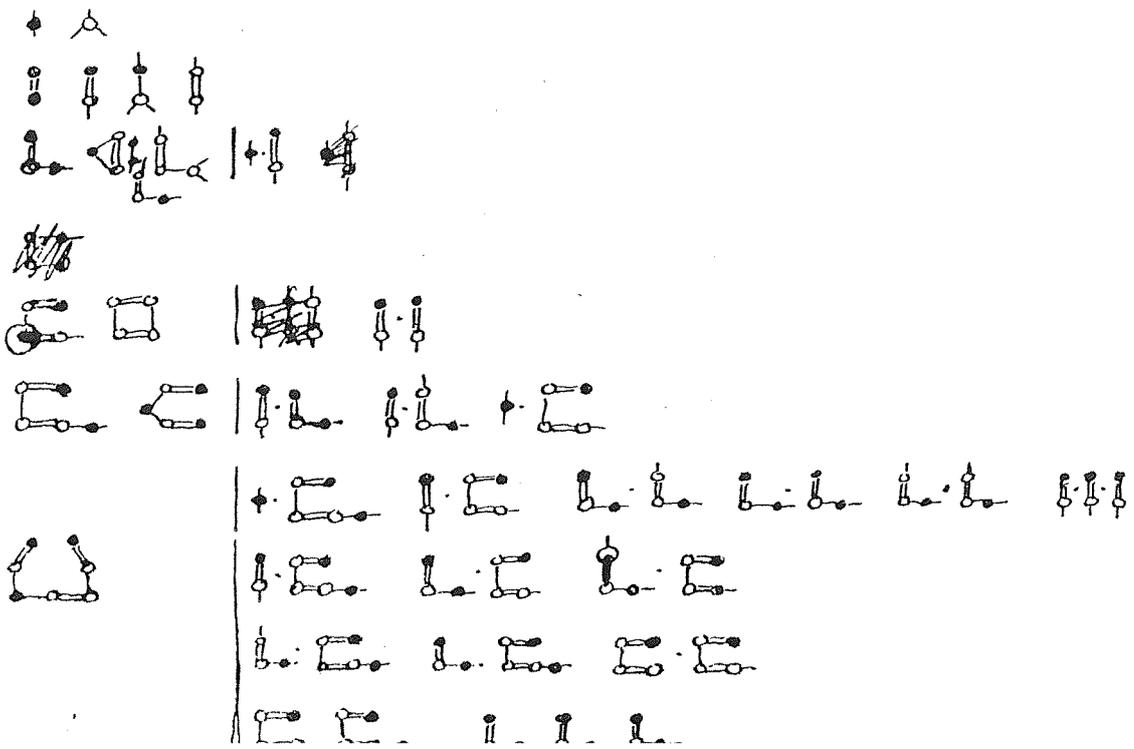
or   $-\mathcal{D}_{ab} \phi$  =  $\phi \mathcal{D}_{ab}$

moreover  +  =  $\supset \mathcal{D}_{ab}$ , which is reduced to zero

we multiply up by the symmetrical quadric  $\phi$

Therefore  +  =  $\phi \mathcal{D}_{ab}$

Hence  =  $-\mathcal{D}_{ab} \phi$  =  $\frac{1}{2} \phi \mathcal{D}_{ab}$



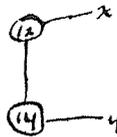
$$123.623 = \Delta | 16$$

$$14x.64x = \overline{74x^2} \cdot (16)$$

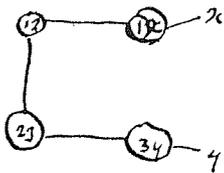
$$123.623.14x.64x = R(x,y)$$

$$589.789 = \Delta | 57$$

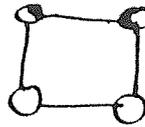
$$1x.12.23.34 = (13).1x.34.27^2 = \overline{13^2.27^2}(xy)$$



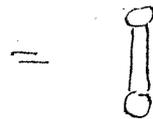
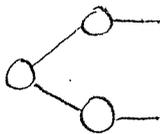
$$= \overline{12^2} \cdot (xy) =$$



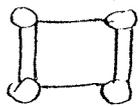
$$= \overline{12^2.34^2} \cdot (xy) =$$



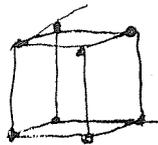
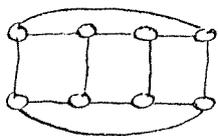
$\phi \subset$



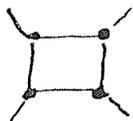
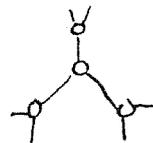
$$= \overline{12^2} \cdot xy$$



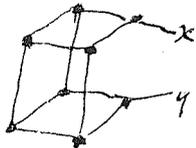
$$= R = \Delta \Delta' = (Rf)$$



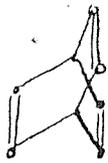
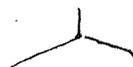
$$= 4R^2$$



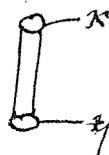
$$= \Delta^2$$



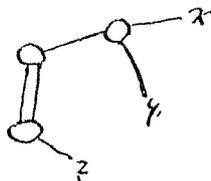
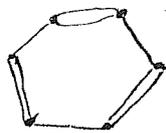
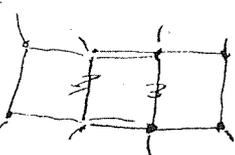
$$= 4R^2(xy)$$



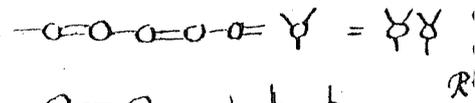
such a poly of  $2n$  sides =  $\Delta^n$



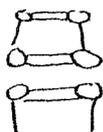
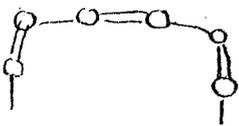
$$= \Delta | xy \text{ at}$$



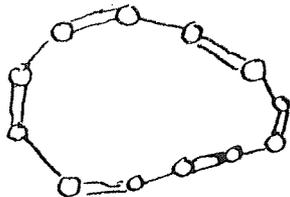
$$= Q | xy z$$



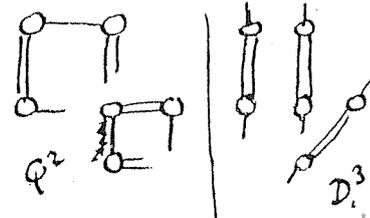
$$\Delta Q =$$



$$RQ =$$



$$= \text{grid of nodes} = R^3$$



$$(a_{11}x_1 + a_{12}x_2)(a_{21}y_1 + a_{22}y_2) - (a_{21}x_1 + a_{22}x_2)(a_{11}y_1 + a_{12}y_2)$$

$$= (a_{11}x_1y_1 + a_{12}x_2y_2)(a_{22} - a_{21}) + x_1y_2(a_{11}a_{22} - a_{21}^2) - x_1y_1(a_{11}a_{12} - a_{12}^2)$$

$$= a_{11}x_1y_1 A (a_{11} - a_{21}) + (xy) D$$

$$12.31.32 = -sD - 2Ds = -3Ds$$

$$(x-y) = \overline{xy} (a_{11} - a_{21}) + (xy) D$$

$$12.31 = \overline{xy} (23) D + 32.5$$

# Binary Sextic

f

H

T

i

A

$\Delta$

$T = T_2 =$

C

l = ; =  $2\Delta + A \frac{1}{3}$

m = or

n = or

these are really equivalent because i is symmetrical

Degree of invariants of ~~cubic~~ quadratic & cubic,  $f_2, f_3$

2, 0

$$D = (ab)^6$$

0, 4

$$R = (\Delta \Delta')^2$$

1, 2

$$E = (a\Delta)^2$$

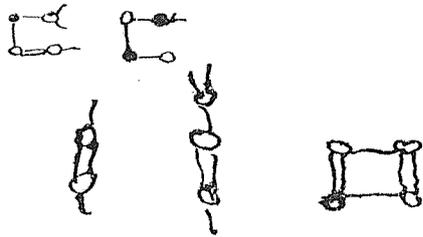
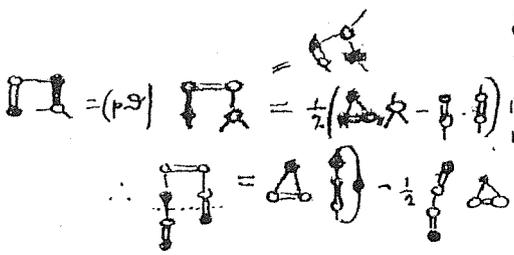
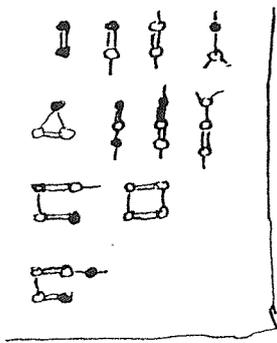
3, 2

$$F = (ap)^2 =$$

$$M = (ap)(ar)$$

There is a series of invariants of a quadratic and any other binary, analogous to the resultant:  $v_3 \cdot (x\alpha \cdot x\beta \cdot y\alpha \cdot y\beta)^k (\overline{x\alpha' y\beta}^{n-k} + \overline{x\beta \cdot y\alpha}^n)$  where  $x, y$  are factors of the quadratic and  $\alpha_2^n = \beta_1^n$  is the form the  $n^{\text{th}}$  order. This vanishes when the condition that one element of the quadric should be the  $k^{\text{th}}$  polar of the other.

Hence in the case of a conic  $C_2$  and a curve  $K_n$  there is a curve of



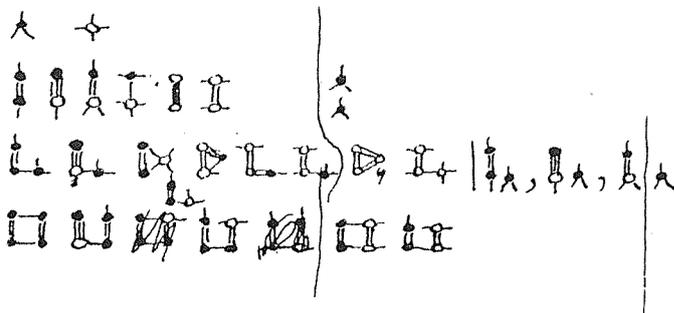
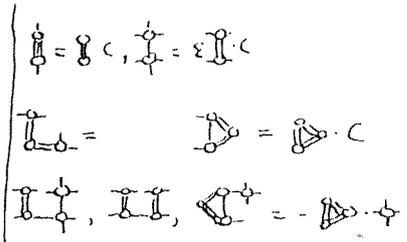
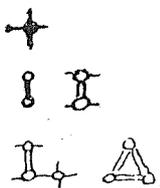
$$\begin{aligned}
 & \times (a_{111} x_1 y_1 a_1 + \dots) \\
 & = \sum_{i,j} (a_{i,j} x_i y_j)^2 \\
 & = \sum_{i,j} c_{ij}^2 + \dots
 \end{aligned}$$

$$a(xu) \cdot b(yu) + a(yu) \cdot b(xu) = a(yu) \cdot b(yu) \cdot |xy|$$

$$a(xu) \cdot b(yu) \cdot c(yz) + a(yu) \cdot b(xu) \cdot c(yz) = -a(yu) \cdot b(yu) \cdot c(xz)$$

$$a(yu) \cdot b(zu) \cdot c(yx) + a(yu) \cdot b(xu) \cdot c(yz) = a(yu) \cdot b(zu) \cdot c(yx) \cdot |xz|$$

$$\therefore a(xu) \cdot b(yu) \cdot c(yz) + c(xy) \cdot a(yu) \cdot b(yz)$$



1/2 hr  
10 min