Classification problems in conformal geometry

Introduction to conformal differential geometry

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Motivation from physics

- GR: null geodesics are conformally invariant
- Maxwell's equations are conformally invariant

g_{ab} = (pseudo-)metric on M, a smooth n-manifold
$\hat{g}_{ab} = \Omega^2 g_{ab}$ = conformally related metric (angles OK)

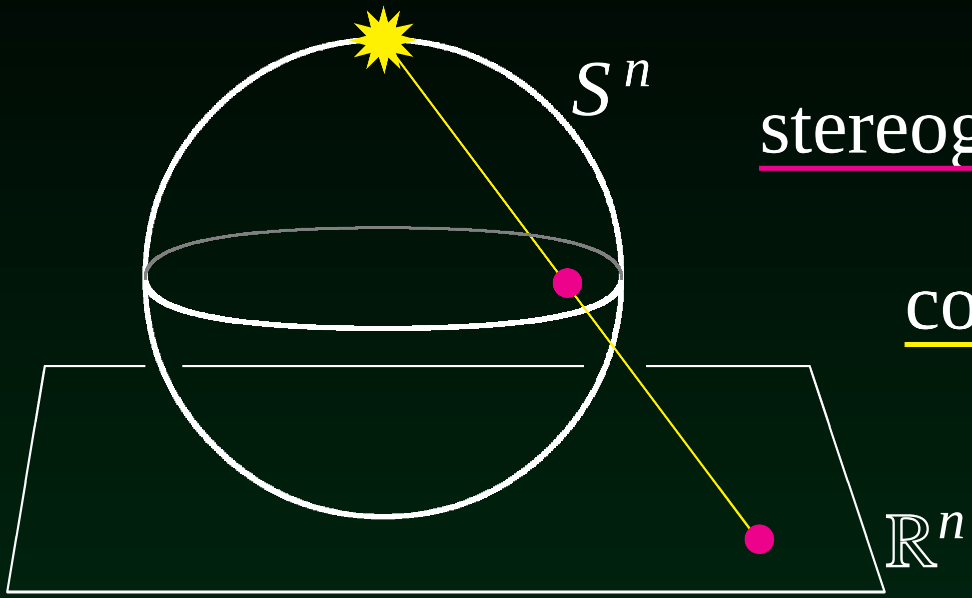
$$g^2 : T^*M \to \mathbb{R} \rightsquigarrow dg^2 \rightsquigarrow X_{g^2} \rightsquigarrow \text{geodesic spray}$$

$$\hat{g}^2 = \Omega^2 g^2 \rightsquigarrow d\hat{g}^2 = \Omega^2 dg^2 + g^2 d\Omega^2 \quad \therefore \boxed{X_{\hat{g}^2}|_{g=0} \propto X_{g^2}|_{g=0}}$$

$g_{ab} \rightsquigarrow \epsilon_{ab\cdots de}$ volume form (e.g. $\epsilon^{ab\cdots de}\epsilon_{ab\cdots de} = n!$)
$\therefore \hat{g}_{ab} = \Omega^2 g_{ab} \implies \hat{\epsilon}_{ab\cdots de} = \Omega^n \epsilon_{ab\cdots de}$
$\therefore \hat{\epsilon}_{ab}{}^{cd} = \epsilon_{ab}{}^{cd}$ when $n = 4$
$\therefore F_{ab} \mapsto *F_{ab} \equiv \epsilon_{ab}{}^{cd} F_{cd}$ is invariant

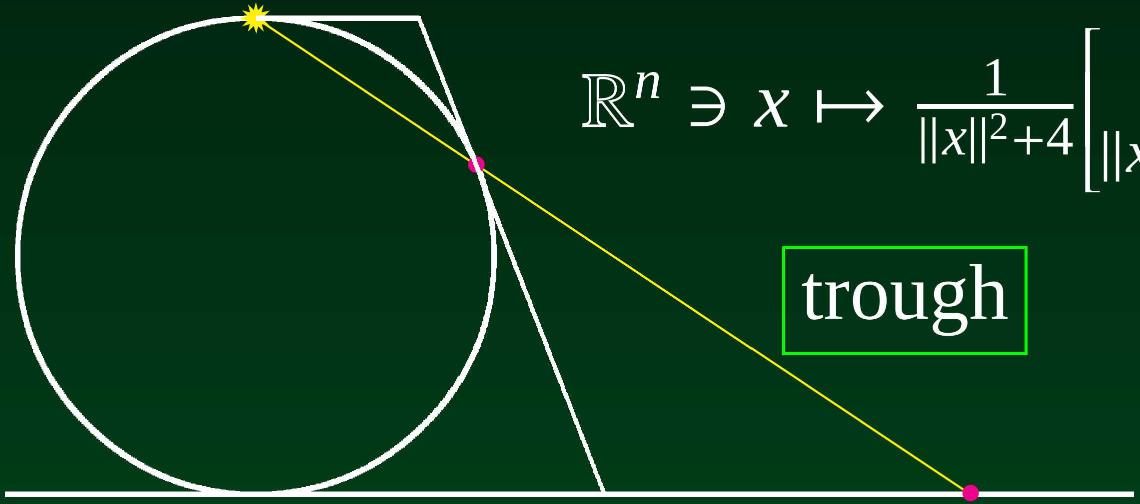
$$\boxed{\begin{aligned} dF &= 0 \\ d{*}F &= 0 \end{aligned}}$$

Motivation from geometry



stereographic projection

conformal

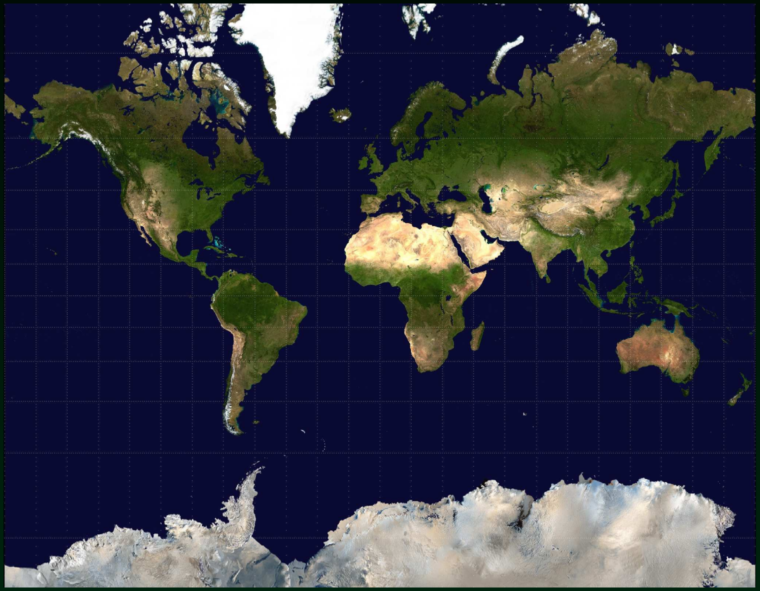


$$\mathbb{R}^n \ni x \mapsto \frac{1}{||x||^2+4}\left[\begin{array}{c}4x\\ ||x||^2-4\end{array}\right] \in S^n$$

trough

Motivation from navigation

- Mercator (Cartographer) 1569



- Wright (Mathematician) 1599

$$S^2 \setminus \{\text{poles}\} \xrightarrow{\text{stereographic}} \mathbb{R}^2 \setminus \{0\} = \mathbb{C} \setminus \{0\} \xrightarrow{\log} \mathbb{C}$$

$$\text{Jac} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \Longleftrightarrow \begin{array}{rcl} u_x & = & v_y \\ v_x & = & -u_y \end{array} \quad \boxed{\text{Cauchy-Riemann}}$$

Euclidean symmetries

X =vector field

$$X = X^1 \frac{\partial}{\partial x_1} + X^2 \frac{\partial}{\partial x_2} + \cdots + X^n \frac{\partial}{\partial x_n} = X^a \nabla_a$$

Infinitesimal Euclidean symmetry: $\underbrace{\mathcal{L}_X \delta_{ab}}_{\text{Lie derivative}} = 0.$

Compute

$$\begin{aligned} \mathcal{L}_X \delta_{ab} &= X^c \nabla_c \delta_{ab} + \delta_{cb} \nabla_a X^c + \delta_{ac} \nabla_b X^c \\ &= \nabla_a X_b + \nabla_b X_a \end{aligned}$$

$$\therefore \ \mathcal{L}_X \delta_{ab} = 0 \iff \boxed{\boxed{\nabla_{(a} X_{b)} = 0}} \ \text{Killing field}$$

Killing fields by prolongation

Killing operator: $\boxed{X_a \mapsto \nabla_{(a}X_{b)}}$

Kernel in flat space: $K_{ab} \equiv \nabla_a X_b$ is skew.

Claim: $\nabla_a K_{bc} = 0$.

$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= 0, \text{ as required.} \end{aligned}$$

Hence, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{array}{rcl} \nabla_a X_b & = & K_{ab} \\ \nabla_a K_{bc} & = & 0 \end{array}}$ Closed!

Conclusion: $\boxed{X_a = s_a + m_{ab}x^b}$ where $m_{ab} = -m_{ba}$.

translations

rotations

Conformal symmetries

$\boxed{\text{trace-free part } \nabla_{(a}X_{b)} = 0}$ conformal Killing field

Rewrite as $\boxed{\nabla_a X_b = K_{ab} + \Lambda\delta_{ab}}$ where K_{ab} is skew.

$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a - \delta_{ab}\nabla_c\Lambda + \delta_{ac}\nabla_b\Lambda \quad \text{so} \end{aligned}$$

$\boxed{\nabla_a K_{bc} = \delta_{ab}Q_c - \delta_{ac}Q_b}$ where $\boxed{\nabla_a\Lambda = -Q_c}$ but

$$\begin{aligned} 0 &= \delta^{ab}(\nabla_d\nabla_a K_{bc} - \nabla_a\nabla_d K_{bc}) \\ &= \delta^{ab}(\delta_{ab}\nabla_d Q_c - \delta_{ac}\nabla_d Q_b - \delta_{db}\nabla_a Q_c + \delta_{dc}\nabla_a Q_b) \\ &= (n-2)\nabla_d Q_c + \delta_{dc}\nabla^a Q_a \quad \text{whence} \end{aligned}$$

$\boxed{\nabla_a Q_b = 0}$ if $n \geq 3$ Closed!!

Conformal symmetries cont'd

Solve
$$\boxed{\begin{array}{rcl} \nabla_a X_b & = & K_{ab} + \Lambda \delta_{ab} \\ \nabla_a K_{bc} & = & \delta_{ab} Q_c - \delta_{ac} Q_b \\ \nabla_a \Lambda & = & -Q_c \\ \nabla_a Q_b & = & 0 \end{array}}$$

$$Q_b = -r_b \qquad \Lambda = \lambda + r^b x_b \qquad K_{bc} = r_b x_c - r_c x_b - m_{bc}$$

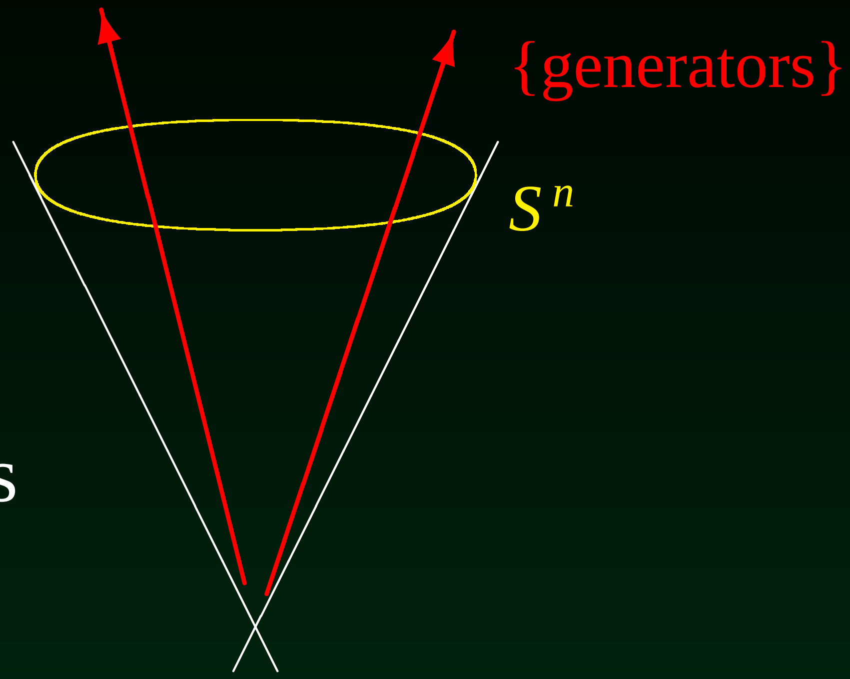
$$\boxed{X_a = s_a + m_{ab} x^b + \lambda x_a + r^b x_b x_a - \tfrac{1}{2} r_a x^b x_b}$$

translation + rotation + dilation + inversion

Integrate the inversions

$$x^a \longmapsto \frac{x^a - \frac{1}{2} r^a ||x||^2}{1 - r_a x^a + \frac{1}{4} ||r||^2 ||x||^2}$$

Conformal group



SO$(n+1,1)$ acts on S^n
by conformal transformations

semisimple

parabolic

$$S^n = \mathrm{SO}(n+1,1)/P$$

flat model of conformal differential geometry

$$\mathbb{R}^n = (\mathrm{SO}(n) \ltimes \mathbb{R}^n)/\mathrm{SO}(n)$$

flat model of Riemannian differential geometry

A simple question on $\mathbb{R}^n$, $n \geq 3$

Question: Which linear differential operators preserve harmonic functions? Answer on $\mathbb{R}^3$:–

Zeroth order $f \mapsto \text{constant} \times f$ 1

First order

$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3$ 3

$x_1\nabla_2 - x_2\nabla_1$ &c. 3

$x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3 \ \boxed{+1/2}$ 1

$(x_1{}^2 - x_2{}^2 - x_3{}^2)\nabla_1 + 2x_1x_2\nabla_2 + 2x_1x_3\nabla_3 + x_1$ 3

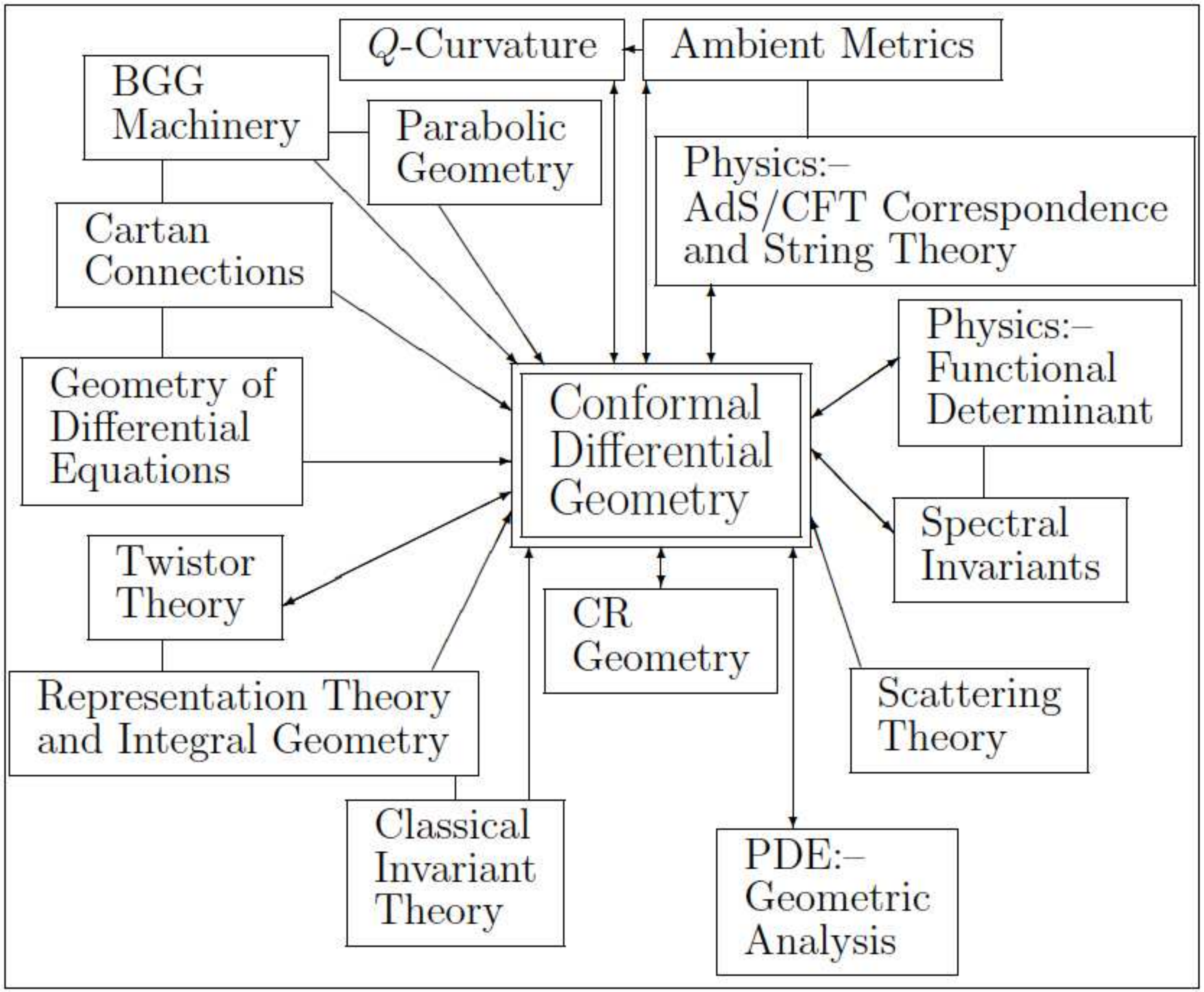
&c.

Dimensions .... $\boxed{10}$

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1$$

Lie Algebra $\cong \mathfrak{so}(4,1) = \boxed{\text{conformal algebra}}$ ← NB!

Surroundings



Q-Curvature

Ambient Metrics

BGG Machinery

Parabolic Geometry

Physics:– AdS/CFT Correspondence and String Theory

Cartan Connections

Physics:– Functional Determinant

Geometry of Differential Equations

Conformal Differential Geometry

Spectral Invariants

Twistor Theory

CR Geometry

Scattering Theory

Representation Theory and Integral Geometry

Classical Invariant Theory

PDE:– Geometric Analysis

Next time

- What about higher order operators preserving harmonic functions? (Beyond first order)
- What about a classification of conformally invariant operators? (Beyond Maxwell)

Further Reading

- C.P. Boyer, E.G. Kalnins, and W. Miller, Jr., Symmetry and separation of variables for the Helmholtz and Laplace equations, Nagoya Math. Jour. **60** (1976) 35–80.
- M.G. Eastwood, Notes on conformal differential geometry, Suppl. Rendi. Circ. Mat. Palermo **43** (1996) 57–76.
- R. Penrose and W. Rindler, Spinors and space-time, vols 1 and 2, Cambridge University Press 1984 and 1986.

THANK YOU

END OF PART ONE