Classification problems in conformal geometry

Methods of Lie theory and geometry

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Recall question from part one

Question: Which linear differential operators on $\mathbb{R}^n$ preserve harmonic functions? Answer on $\mathbb{R}^3$:–

Zeroth order $\quad f \mapsto \text{constant} \times f$ $\quad 1$

First order

$$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3 \qquad 3$$

$$x_1\nabla_2 - x_2\nabla_1 \quad \text{\&c.} \qquad 3$$

$$x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3 \boxed{+1/2} \qquad 1$$

$$({x_1}^2 - {x_2}^2 - {x_3}^2)\nabla_1 + 2x_1x_2\nabla_2 + 2x_1x_3\nabla_3 + x_1 \qquad 3$$

&c.

Dimensions . . . . $\boxed{10}$

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1$$

Lie Algebra $\cong \mathfrak{so}(4,1) =$ $\boxed{\text{conformal algebra}}$ $\leftarrow$ NB!

Second order

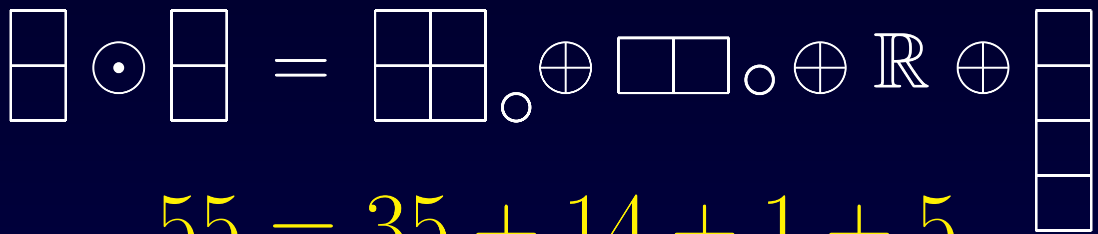
Boyer-Kalnins-Miller (1976)

Extras: $\propto$ Laplacian $(f \mapsto h\Delta f$ for any smooth $h)$

plus a 35-dim$^{\ell}$ family of new ones!

$$\{\mathcal{D}_1, \mathcal{D}_2\} \equiv \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1$$

$\bigodot^2 \mathfrak{so}(4,1) = ?$ $\quad \mathrm{dim} = 10 \times 11/2 = 55$

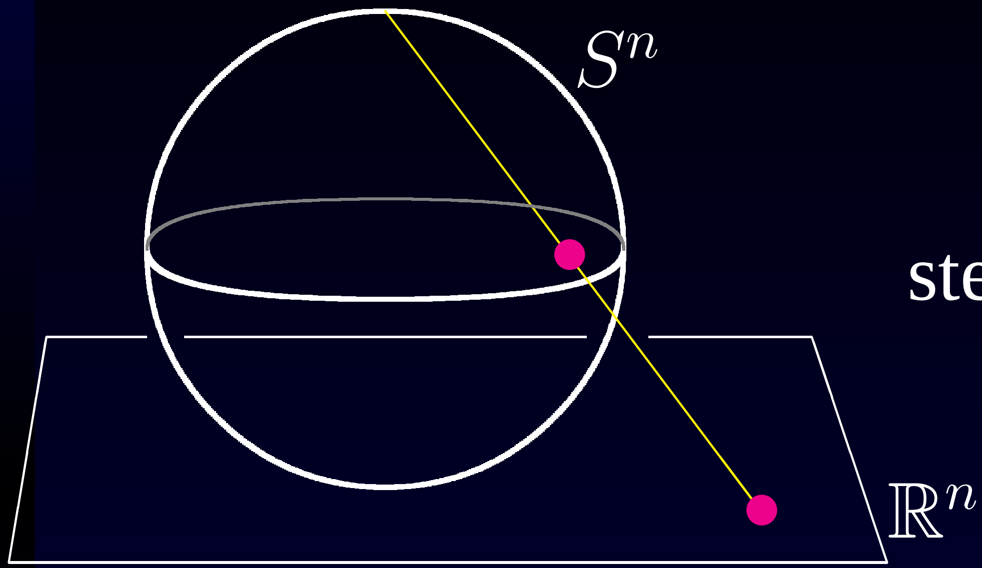


$$55 = 35 + 14 + 1 + 5$$

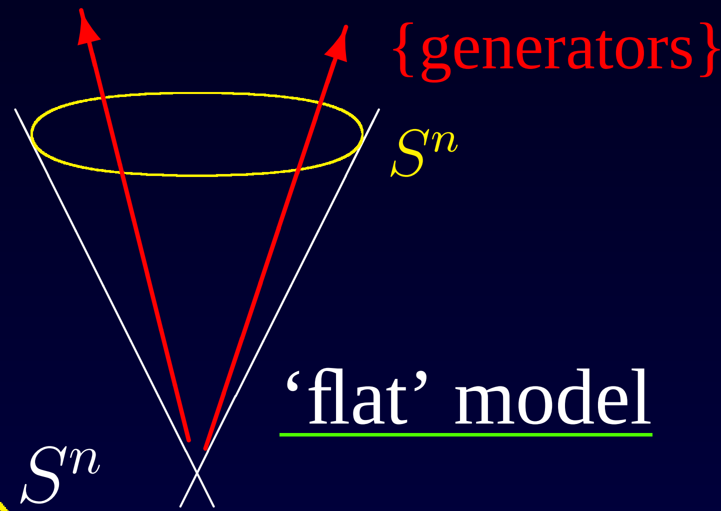
Separation of variables (Bôcher, Bateman, ...).

Third order...?

Recall flat conformal geometry



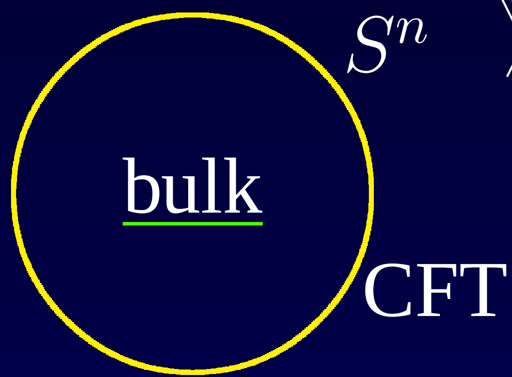
stereographic projection



Action of $\mathrm{SO}(n+1,1)$ on S^n
by conformal transformations

'flat' model

Beltrami model of
hyperbolic space



Conformal Laplacian **Dirac 1935**

$$r \equiv {x_1}^2 + \cdots {x_n}^2 + {x_{n+1}}^2 - {x_{n+2}}^2$$

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2} + \frac{\partial^2}{\partial {x_{n+1}}^2} - \frac{\partial^2}{\partial {x_{n+2}}^2}$$

f on null cone $\subset \mathbb{R}^{n+2}$ homogeneous of degree $w \rightsquigarrow$

- ambiently extend to $\tilde{f}$ of degree w
- freedom $\tilde{f} \mapsto \tilde{f} + rg$ for g of degree $w - 2$
- calculate: $\tilde{\Delta}(rg) = r\tilde{\Delta}g + 2(n + 2w - 2)g$

$w = 1 - n/2 \Rightarrow f \mapsto (\tilde{\Delta}\tilde{f})|_{r=0}$ is invariantly defined.

On $\mathbb{R}^n$ it's Δ

On S^n it's $\Delta - \frac{n-2}{4(n-1)}R$

AdS/CFT

Fefferman-Graham ‘ambient’ metric

Symmetries of Δ

$\mathcal{D}$ a symmetry $\iff$ $\Delta\mathcal{D} = \delta\Delta$ for some δ.

trivial example: $\mathcal{D} = \mathcal{P}\Delta$ for any $\mathcal{P}$

equivalence: $\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$

$\mathbb{R}^n \rightsquigarrow \mathcal{A}_n \equiv$ algebra of symmetries

under composition
up to equivalence

Write $\mathcal{D} = V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_d$ + lower order terms

symbol

normalise w.l.g. to be trace-free

Theorems

- $\mathcal{D}$ a symmetry $\Rightarrow$ trace-free part of $\nabla^{(a}V^{bc\cdots d)} = 0$
  - Easy ☺
- On $\mathbb{R}^n$, such a conformal Killing tensor $V^{bc\cdots d} \rightsquigarrow \mathcal{D}_V$
  - Not So Easy ☹

$\mathcal{D}_V$ is a canonically associated symmetry of the form

$$\mathcal{D}_V = V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_d + \text{lower order terms.}$$

- E.g. First order

$$\mathcal{D}_V f = V^a\nabla_a f + \tfrac{n-2}{2n}(\nabla_a V^a)f$$

- E.g. Second order

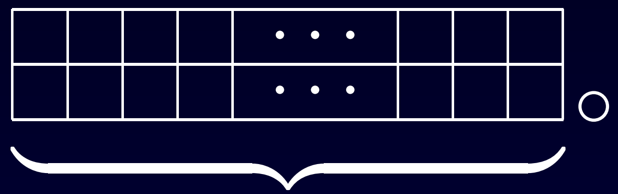
$$\mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f + \tfrac{n}{n+2}(\nabla_a V^{ab})\nabla_b f + \tfrac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b V^{ab})f$$

Ingredients of proof

- We can solve the conformal Killing tensor equation

$$\nabla^{(a}V^{bc\cdots d)} = g^{(ab}\lambda^{c\cdots d)}$$

on $\mathbb{R}^n$ by prolongation (use Lie algebra cohomology):–



w.r.t. $\mathfrak{so}(n+1,1)$.

# of columns = # of indices on $V^{bc\cdots d}$

$$\begin{aligned} \text{E.g. } V^b &= s^b + m^{bc}x_c + \lambda x^b + r^c x_c x^b - \tfrac{1}{2}x^c x_c r^b \\ &= \text{translation} + \text{rotation} + \text{dilation} + \text{inversion.} \end{aligned}$$

- Use ‘ambient methods’ to construct $\mathcal{D}_V$.

Algebra structure

As a vector space

$$\mathcal{A}_n = \bigoplus_{s=0}^{\infty} \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & \cdots & & & \\ \hline & & & & \cdots & & & \\ \hline\end{array}}_{s} .$$

Question: What about the algebra structure?

Theorem

$$\mathcal{A}_n = \frac{\bigotimes \mathfrak{so}(n+1,1)}{\left(X \otimes Y - X \odot Y - \frac{1}{2}[X,Y] + \frac{n-2}{4n(n+1)}\langle X,Y\rangle\right)}$$

Cartan Lie Killing

Equivalently,

$$\mathcal{A}_n = \mathfrak{U}(\mathfrak{so}(n+1,1))/\text{Joseph Ideal}.$$

Proof of algebra structure

Calculate by ambient means that

$$\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \circledcirc Y} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} - \tfrac{n-2}{4n(n+1)}\mathcal{D}_{\langle X,Y\rangle}$$

and use properties of Cartan product (due to Kostant).

Remark: simple Lie algebra $= \mathfrak{g} \neq \mathfrak{sl}(2,\mathbb{C}) \Rightarrow$

$$\dim \frac{\bigotimes \mathfrak{g}}{(X \otimes Y - X \circledcirc Y - \frac{1}{2}[X,Y] - \lambda\langle X,Y\rangle)} = \infty$$

for precisely one value of λ (Braverman and Joseph)

$\rightsquigarrow$ graded algebra $\bigoplus_{s=0}^{\infty} \circledcirc^s \mathfrak{g}$.

Conformal Laplacian cont'd

Know: $\Delta - \frac{n-2}{4(n-1)}R$ is invariant under $\mathrm{SO}(n+1,1)$.

Better: $\Delta - \frac{n-2}{4(n-1)}R$ is 'conformally invariant'.

Meaning:

$$g_{ab} \rightsquigarrow \hat{g}_{ab} = \Omega^2 g_{ab} \quad \nabla_a \rightsquigarrow \hat{\nabla}_a \quad R_{abcd} \rightsquigarrow \hat{R}_{abcd} \quad R \rightsquigarrow \hat{R}$$

EG: $\hat{R} = \Omega^{-2}\left(R - (n-1)(2\nabla^a\Upsilon_a + (n-2)\Upsilon^a\Upsilon_a)\right)$
where $\Upsilon_a \equiv \nabla_a \log\Omega$

$$\left(\hat{\Delta} - \frac{n-2}{4(n-1)}\hat{R}\right)\Omega^{1-\frac{n}{2}}f = \Omega^{-1-\frac{n}{2}}\left(\Delta - \frac{n-2}{4(n-1)}R\right)f$$

$$L \equiv \Delta - \frac{n-2}{4(n-1)}R \qquad \hat{L}\hat{f} = \widehat{Lf} \qquad \hat{f} = \Omega^{1-\frac{n}{2}}f$$

L invariantly defined between suitable line bundles.

Operators on the three-sphere

A complete list of $\mathrm{SO}(4,1)$-invariant linear differential operators between irreducible tensor bundles

- Standard (with suitable conformal weights)

$$\bigodot_\circ^b \Lambda^1 \xrightarrow{\nabla^{a+1}} \bigodot_\circ^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{2b+1}} \bigodot_\circ^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{a+1}} \bigodot_\circ^b \Lambda^1$$

for $a, b \in \mathbb{Z}_{\geq 0}$ $(a = b = 0 \rightsquigarrow$ de Rham complex)

- Non-standard

$$\bigodot_\circ^b \Lambda^1[a+2b] \xrightarrow{\nabla^{2a+2b+3}} \bigodot_\circ^b \Lambda^1[-a-3]$$

for $a + 1/2, b \in \mathbb{Z}_{\geq 0}$ $(a = -1/2, b = 0 \rightsquigarrow$ Laplacian)

Proof is by algebra (Lie theory and Verma modules)

Theorem All these operators have conformally invariant 'curved analogues.'

Conformal-to-Einstein operator

Let $\mathrm{P}_{ab} \equiv \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R g_{ab}\right)$. Then

$$\sigma \stackrel{D}{\longmapsto} \text{trace-free part of}(\nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma)$$

is conformally invariant when $\hat{\sigma} = \Omega\sigma$ $(a = 1, b = 0)$

Geometric meaning where $\sigma \neq 0$ (LeBrun 1985)

$$D\sigma = 0 \iff \sigma^{-2} g_{ab} \text{ is an Einstein metric}$$

Prolong

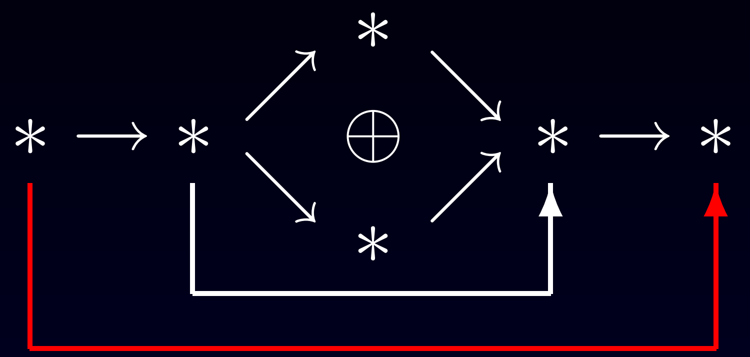
$$D\sigma = 0 \iff \begin{aligned} \nabla_a \sigma - \mu_a &= 0 \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho &= 0 \\ \nabla_a \rho - \mathrm{P}_a{}^b \mu_b &= 0 \end{aligned}$$

Cartan connection

$\rightsquigarrow$ Curved translation principle

Beware the four-sphere

Pattern



cf. de Rham
Maxwell

Theorem Most of these operators have conformally invariant 'curved analogues.'

Standard ✓ Non-standard ☹

Δ	✓	Bateman, Yamabe, et alia
Δ^2	✓	Paneitz, Riegert, Eastwood & Singer
Δ^3	✗	Graham
$\Delta^{\geq 4}$	✗	Gover & Hirachi
general	☹	cf. Eastwood & Slovák

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THANK YOU

THE END