Invariant differential operators and exterior differential systems II

Symmetries, Lie algebra cohomology, and the BGG machinery: examples

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Tableau from Lecture I

Recall tableau $A \subseteq W \otimes V^*$ and

$$\begin{tikzcd} A \otimes V^* \arrow[r, hook] \arrow[rrr, bend right, "\delta"'] & W \otimes V^* \otimes V^* \arrow[rr, "\mathbb{1}\otimes\wedge"] & & W \otimes \Lambda^2 V^* \end{tikzcd}$$

$\ker\delta \equiv A^{(1)} = A \otimes V^* \cap W \otimes \bigodot^2 V^*$
$\operatorname{coker}\delta \equiv H^{0,2}(A)$

Prefer $A \subseteq V^* \otimes W$ and $\partial : V^* \otimes A \to \Lambda^2 V^* \otimes W$
Reason $D : E \to F$ first order $\rightsquigarrow$ symbol

$$\sigma(D) : \Lambda^1 \otimes E \to F$$

Suppose surjective and let $\boxed{K \equiv \ker\sigma(D)} \sim A$

Killing operator from Lecture I

$$D : E \to F \qquad \rightsquigarrow K \equiv \ker \sigma(D) \subset \Lambda^1 \otimes E$$
$$\rightsquigarrow \partial : \Lambda^1 \otimes K \to \Lambda^2 \otimes E$$

$\boxed{\text{EG}}$ Killing equation $\mathcal{L}_X g_{ab} = 2\nabla_{(a}X_{b)} = 0$

Killing operator $\nabla : \Lambda^1 \to \bigodot^2 \Lambda^1$

$$X_a \mapsto \nabla_{(a}X_{b)}$$

$$\sigma(\nabla) : \Lambda^1 \otimes \Lambda^1 \xrightarrow{\odot} \bigodot^2 \Lambda^1 \implies K = \Lambda^2 \subset \Lambda^1 \otimes \Lambda^1$$

$\boxed{\text{NB}}$ $\partial : \Lambda^1 \otimes K = \Lambda^1 \otimes \Lambda^2 \xrightarrow{\cong} \Lambda^2 \otimes \Lambda^1$

$$Z_{abc} \longmapsto Z_{[ab]c}$$

$$Y_{abc} + Y_{cab} + Y_{cba} = 4Y_{a[bc]} - 3Y_{[abc]} \longleftarrow Y_{abc}$$

$\boxed{\text{Cf}}$ existence and uniqueness of Levi-Civita connection

Prolongation of Killing revisited

$$\nabla_{(a}X_{b)} = 0 \iff \nabla_a X_b = Z_{ab} \ \text{ for } \ Z_{ab} = Z_{[ab]}$$

But then,
$$\begin{aligned} \nabla_a Z_{bc} &= \nabla_c Z_{ba} - \nabla_b Z_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= R_{bc}{}^d{}_a X_d \end{aligned}$$

Therefore, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{aligned} \nabla_a X_b &= Z_{ab} \\ \nabla_a Z_{bc} &= R_{bc}{}^d{}_a X_d \end{aligned}}$

Hence, Killing fields $\leftrightarrow$ covariant constant sections of $V \equiv \Lambda^1 \oplus \Lambda^2$ with connection

$$\begin{bmatrix} X_b \\ Z_{bc} \end{bmatrix} \xmapsto{\nabla_a} \begin{bmatrix} \nabla_a X_b - Z_{ab} \\ \nabla_a Z_{bc} - R_{bc}{}^d{}_a X_d \end{bmatrix}$$

Coupled de Rham sequence

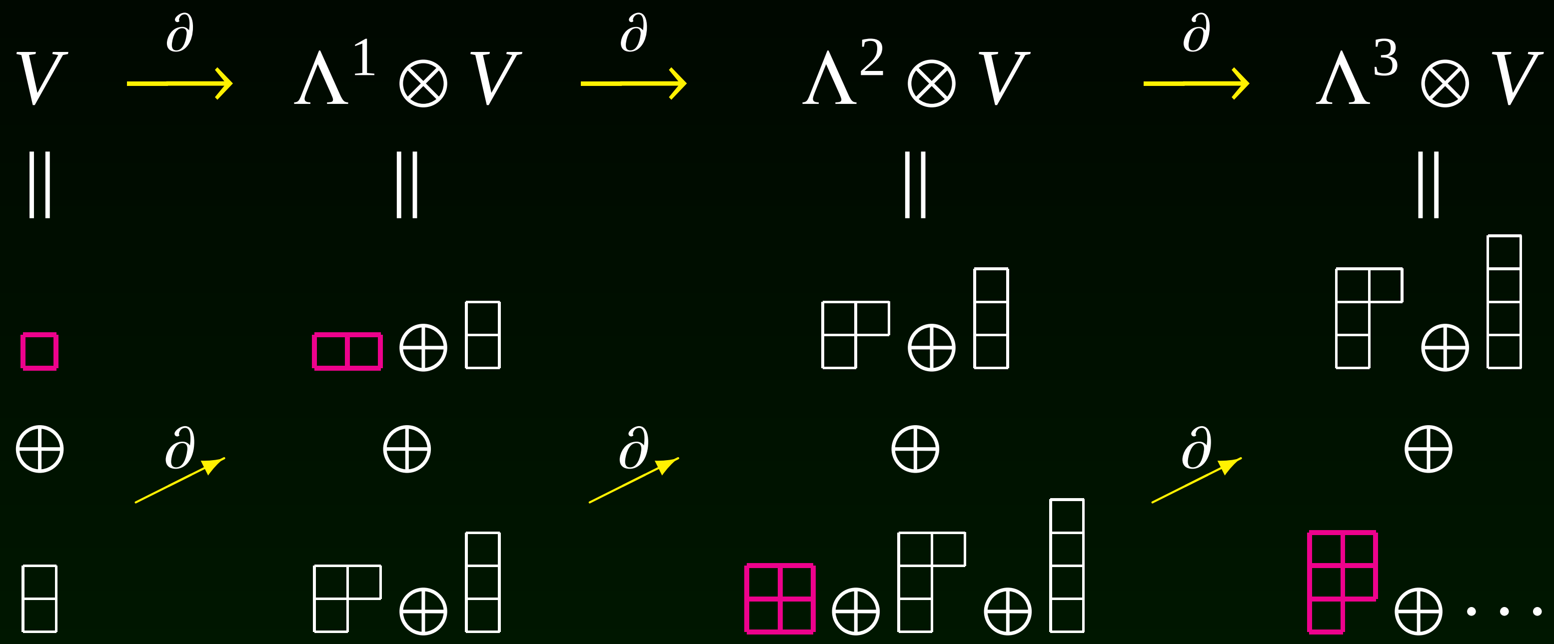
$$\begin{bmatrix} X_b \\ Z_{bc} \end{bmatrix} \stackrel{\nabla}{\longmapsto} \begin{bmatrix} \nabla_a X_b - Z_{ab} \\ \nabla_a Z_{bc} - R_{bc}{}^{d}{}_{a} X_d \end{bmatrix}$$

$$\begin{tikzcd}
\Lambda^1 \arrow[r] \arrow[ddr] & \Lambda^1 \otimes \Lambda^1 \\
\oplus & \oplus \\
\Lambda^2 \arrow[r] \arrow[uur] & \Lambda^1 \otimes \Lambda^2
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, "\nabla"] \arrow[d, equal] & \Lambda^1 \otimes V \arrow[r, "\nabla"] \arrow[d, equal] & \Lambda^2 \otimes V \arrow[r, "\nabla"] \arrow[d, equal] & \Lambda^3 \otimes V \arrow[r, "\nabla"] \arrow[d, equal] & \cdots \\
\Lambda^1 & \Lambda^1 \otimes \Lambda^1 & \Lambda^2 \otimes \Lambda^1 & \Lambda^3 \otimes \Lambda^1 & \cdots \\
\oplus & \oplus & \oplus & \oplus & \\
\Lambda^2 \arrow[uur] & \Lambda^1 \otimes \Lambda^2 \arrow[uur, "\boxed{\text{NB}}"] & \Lambda^2 \otimes \Lambda^2 \arrow[uur] & \Lambda^3 \otimes \Lambda^2 \arrow[uur] & \cdots
\end{tikzcd}$$

$$\Lambda^p \otimes \Lambda^2 \ni \underset{\substack{\| \\ Z_{[a\cdots b][cd]}}}{Z_{a\cdots bcd}} \stackrel{\partial}{\longmapsto} Z_{[a\cdots bc]d} \in \Lambda^{p+1} \otimes \Lambda^1$$

Decompose into irreducibles



Lie algebra cohomology! (Kostant 1961)

Suspend disbelief

EG: $\ker : \Lambda^2 \otimes \Lambda^2 \xrightarrow{\partial} \Lambda^3 \otimes \Lambda^1$

$= \{Z_{abcd} = Z_{[ab][cd]} \text{ s.t. } Z_{[abc]d} = 0\}$

$= \{$Riemann curvature tensors$\}$!

Curvature (Corollary 3.2.3 revisited)

$$\begin{array}{ccccc}
V & \xrightarrow{\nabla} & \Lambda^1 \otimes V & \xrightarrow{\nabla} & \Lambda^2 \otimes V \\
\| & & \| & & \| \\
\Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 \\
\oplus & & \oplus & & \oplus \\
\Lambda^2 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^2
\end{array}$$

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix} X_c \\ Z_{cd} \end{bmatrix} = \begin{bmatrix} 0 \leftsquigarrow \text{by design} \\ R \bowtie Z + (\nabla R) \bowtie X \end{bmatrix}$$

$$\boxed{2R_{ab}{}^e{}_{[c}Z_{d]e} + 2R_{cd}{}^e{}_{[a}Z_{b]e}}$$

$$\boxed{(\nabla_b R_{cd}{}^e{}_a)X_e - (\nabla_a R_{cd}{}^e{}_b)X_e}$$

Flat $\iff$ $R_{abcd} = \lambda(g_{ac}g_{bd} - g_{bc}g_{ad})$

$\iff$ constant curvature

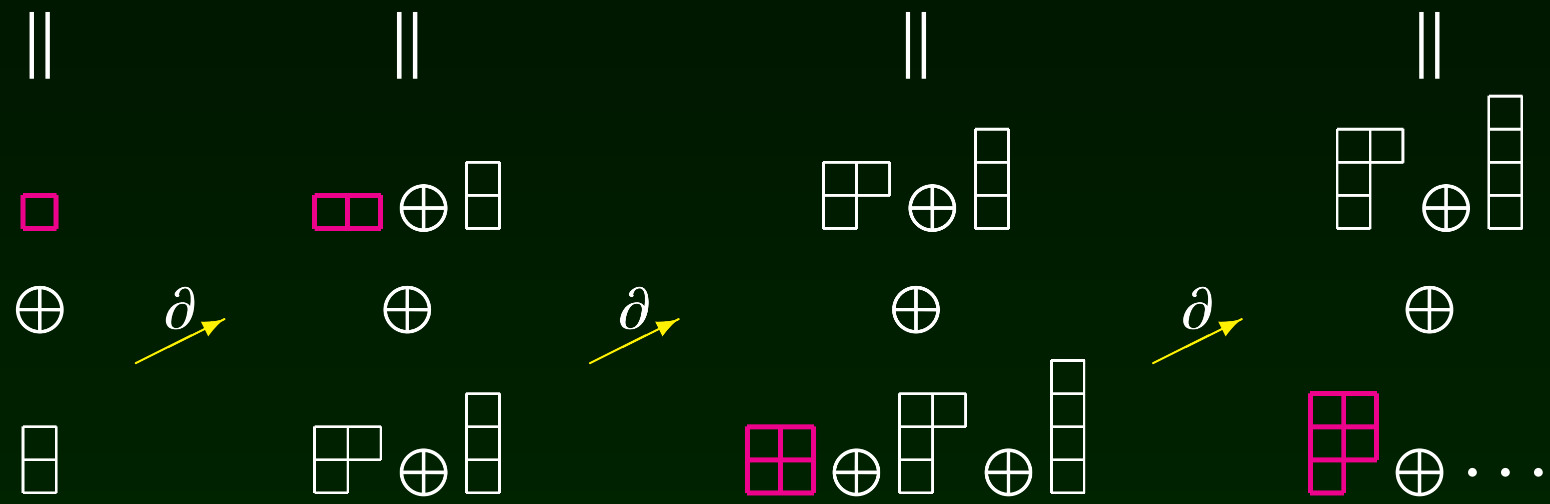
Maximal symmetry

Spherical	$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$	$\mathrm{SO}(n+1)$
Euclidean	$R_{abcd} = 0$	$\mathrm{SO}(n) \ltimes \mathbb{R}^n$
Hyperbolic	$R_{abcd} = -g_{ac}g_{bd} + g_{bc}g_{ad}$	$\mathrm{SO}(n,1)$

$$\dim = \dim \Lambda^1 + \dim \Lambda^2 = n + n(n-1)/2 = n(n+1)/2$$

Diagram chasing in the constant curvature case. . .

$$V \xrightarrow{\nabla} \Lambda^1 \otimes V \xrightarrow{\nabla} \Lambda^2 \otimes V \xrightarrow{\nabla} \Lambda^3 \otimes V$$



leads to. . .

BGG resolutions

The locally exact complexes

Riemannian deformation

$$\ydiagram{1} \xrightarrow{\nabla} \ydiagram{2} \xrightarrow{\nabla^{(2)}} \ydiagram{2,2} \xrightarrow{\nabla} \ydiagram{2,2,1} \xrightarrow{\nabla} \cdots$$

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots$$

are Bernstein-Gelfand-Gelfand resolutions.

de Rham

$$n\text{-sphere} = \mathrm{SL}(n+1,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} \lambda & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}, \lambda > 0 \right\} = G/P,$$

where G is semisimple and P is parabolic.
The round n-sphere is projectively flat (Thales).

Lie algebra cohomology

$\boxed{\mathfrak{u} = \text{Lie algebra}}$ $\boxed{\mathbb{V} = \mathfrak{u}\text{-module}}$

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{u}, \mathbb{V}) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2\mathfrak{u}, \mathbb{V}) \xrightarrow{\partial} \cdots$$

$$\partial v(X) = Xv \qquad \partial\phi(X \wedge Y) = \phi([X,Y]) - X\phi(Y) + Y\phi(X) \qquad \cdots$$

$$\rightsquigarrow \boxed{H^p(\mathfrak{u}, \mathbb{V})}$$

$$\mathfrak{sl}(n+1, \mathbb{R}) \ni \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array}\right]$$

$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Let $\mathfrak{u} = \mathfrak{g}_{-1}$ ($\Rightarrow \mathfrak{u}^* = \mathfrak{g}_1$)

Let $\mathbb{V} = \Lambda^2\mathbb{R}^{n+1}|_{\mathfrak{g}_{-1}}$

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2\mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial}$$

Geometric import

Kostant's Bott-Borel-Weil Theorem $\Longrightarrow$

$H^p(\mathfrak{g}_{-1}, \mathbb{V}) = \ydiagram{1}, \ydiagram{2}, \ydiagram{2,2}, \cdots$ as $\mathrm{SL}(n, \mathbb{R})$-modules

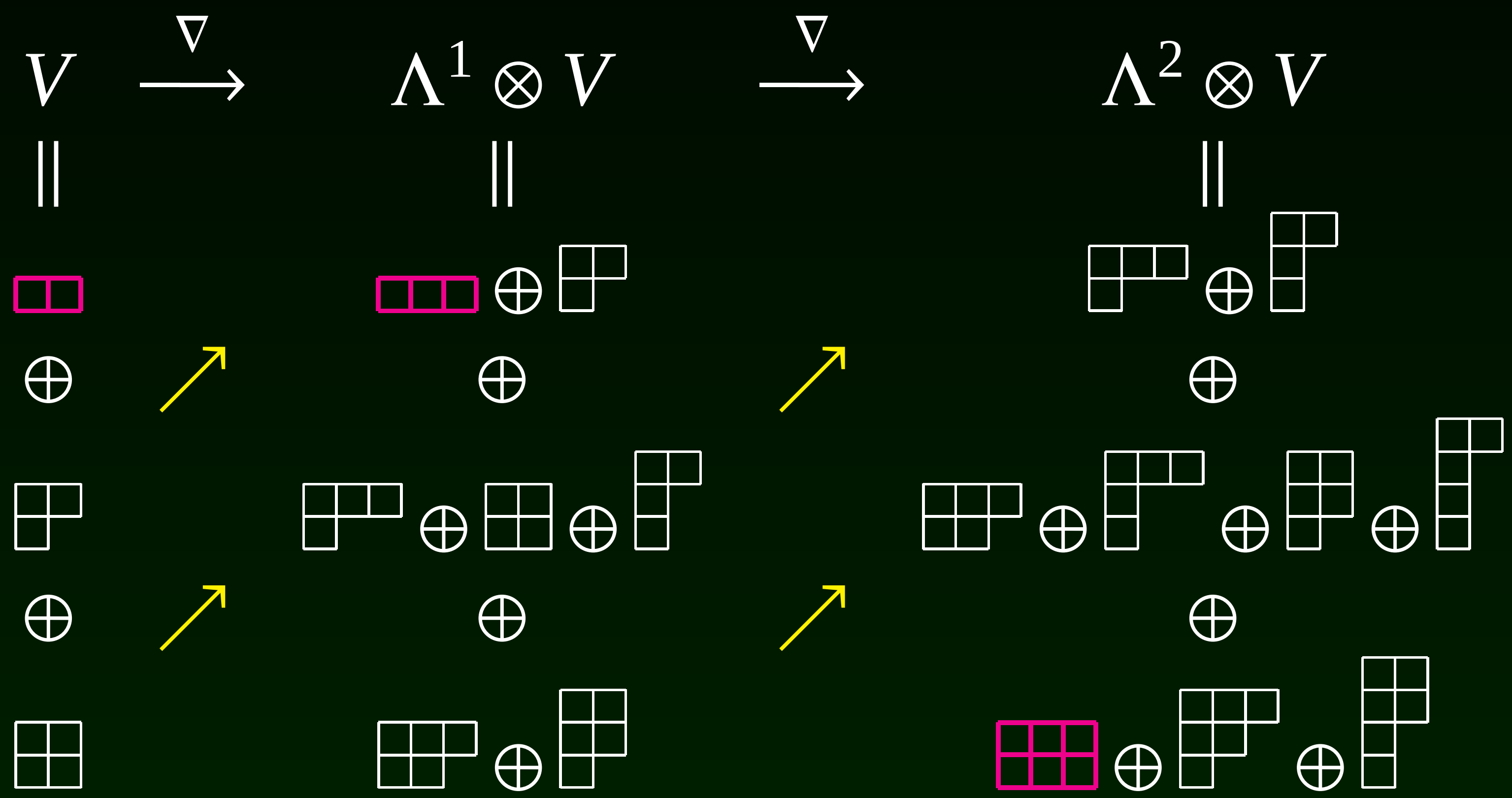
$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^2\mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^3\mathfrak{g}_1 \otimes \mathbb{V} \\
 & & \wr & & \wr & & \wr & & \wr \\
0 & \to & V & \xrightarrow{\partial} & \Lambda^1 \otimes V & \xrightarrow{\partial} & \Lambda^2 \otimes V & \xrightarrow{\partial} & \Lambda^3 \otimes V \\
 & & \| & & \| & & \| & & \| \\
 & & \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 \\
 & & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus \\
 & & \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2
\end{array}
$$

$\boxed{\text{Previously suspected tensor identities are justified}}$!!

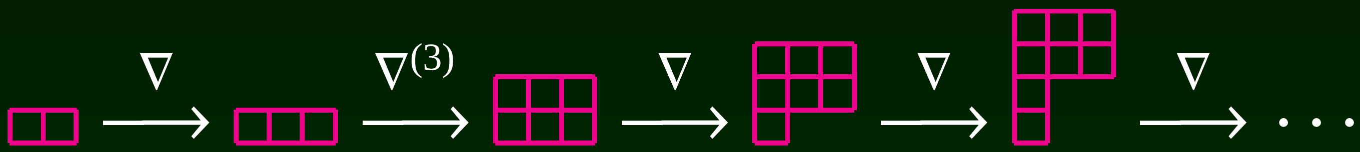
Higher Killing operators

prolong using Lie algebra cohomology

$$X_{bc} = X_{(bc)} \mapsto \nabla_{(a}X_{bc)}$$



BGG



General (projective) case

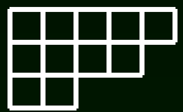
Recall Killing $\nabla : \Lambda^1 \to \odot^2 \Lambda^1$

Higher Killing $\nabla : \odot^\ell \Lambda^1 \to \odot^{\ell+1} \Lambda^1$

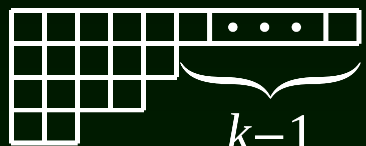
Generalised Killing $\nabla^k : \odot^\ell \Lambda^1 \to \odot^{k+\ell} \Lambda^1$

$\vdots$

$$\boxed{\nabla^k : E \to \odot^k \Lambda^1 \circledcirc E}$$

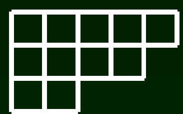
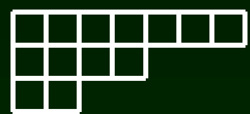
 $\xrightarrow{\nabla^k}$ 

k

$\mathbb{V} =$  $(\mathbb{R}^{n+1})$ as $\mathrm{SL}(n+1,\mathbb{R})$-module

$k-1$

prolong using Lie algebra cohomology $H^p(\mathfrak{g}_{-1}, \mathbb{V})$ science ~~art form~~ 

$\boxed{\text{EG}}$ $\dim \ker \nabla^2 :$  $\to$  $\leq \frac{(n-2)(n-1)^2 n^2 (n+1)^3 (n+2)^3 (n+3)^2 (n+4)^2 (n+5)(n+6)}{261273600}$

Sharp dimension bound

After six prolongations

$$\boxed{\frac{(n-2)(n-1)^2 n^2 (n+1)^3 (n+2)^3 (n+3)^2 (n+4)^2 (n+5)(n+6)}{261273600}}$$

Projective Killing fields

$$X^a \text{ such that } (\underbrace{\nabla_{(a}\nabla_{b)}X^c + \mathrm{P}_{ab}X^c}_{\text{first BGG operator}} + \underbrace{W_{d(a}{}^c{}_{b)}}_{\text{Weyl curvature}}X^d)_\circ = 0$$

where $\mathrm{P}_{ab} = \frac{1}{n-1}R_{ab}$ is the Rho-tensor (when special)

Prolong

$$\begin{array}{rcl} \nabla_a X^c & = & Y_a{}^c \\ \nabla_a Y_b{}^c - R_{ab}{}^c{}_d X^d + \mathrm{P}_{ab}X^c + W_{d(a}{}^c{}_{b)}X^d & = & Z_a\delta_b{}^c + Z_b\delta_a{}^c \\ \nabla_a Z_b & = & \cdots \quad \text{closes!} \end{array}$$

Connection

$$\begin{bmatrix} X^c \\ Y_b{}^c \\ Z_b \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a X^c - Y_a{}^c \\ \nabla_a Y_b{}^c - (R \bowtie X)_{ab}{}^c - Z_a\delta_b{}^c - Z_b\delta_a{}^c \\ \nabla_a Z_b - \cdots \end{bmatrix}$$

Conformal Killing fields

$$\boxed{\text{the trace-free part of } \nabla_{(a}\sigma_{b)} = 0}$$

$$\begin{aligned}
&\nabla_a\sigma_b = \mu_{ab} + \nu g_{ab} \quad \text{where } \mu_{ab} \text{ is skew}\\
&\nabla_a\nu = -\rho_a - \mathrm{P}_a{}^b\sigma_b\\
&\nabla_a\mu_{bc} = g_{ab}\rho_c - g_{ac}\rho_b - \mathrm{P}_{ab}\sigma_c + \mathrm{P}_{ac}\sigma_b + \boxed{C_{bc}{}^d{}_a\sigma_d}\\
&\nabla_a\rho_b = \mathrm{P}_a{}^c\mu_{bc} + \mathrm{P}_{ab}\nu + \boxed{Y^c{}_{ab}\sigma_c}
\end{aligned}$$

where

Cartan connection. . .

$$\begin{aligned}
&R_{abcd} = C_{abcd} + \mathrm{P}_{ac}g_{bd} - \mathrm{P}_{bc}g_{ad} + \mathrm{P}_{bd}g_{ac} - \mathrm{P}_{ad}g_{bc}\\
&Y_{abc} = \nabla_a\mathrm{P}_{bc} - \nabla_b\mathrm{P}_{ac}
\end{aligned}$$

Sample (symplectic) identity

$\boxed{J_{ab} = \text{non-degenerate symplectic form}}$

Suppose T_{abcd} has the following symmetries

- $T_{abcd} = T_{[ab][cd]}$,
- $T_{[abc]d} = J_{[ab}\psi_{c]d}$ for some tensor ψ_{cd}.

Then there are unique tensors

- ρ_{ab},
- $\tau_{ab} = \tau_{[ab]}$,
- $X_{abcd} = X_{[ab][cd]}$ with $X_{[abc]d} = 0$ and $J^{ab}X_{abcd} = 0$,

such that

$$T_{abcd} = X_{abcd} + J_{c[a}\rho_{b]d} - J_{d[a}\rho_{b]c} - J_{cd}\rho_{[ab]} + J_{ab}\tau_{cd}.$$

Easy algorithms!

Many Happy Returns Robert!