Conformal differential geometry

and its interaction with representation theory

Conformally invariant differential operators

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Examples

- Maxwell in dimension four
- Laplacian in dimension two $(\Delta = 4\partial^2/\partial z\partial\bar{z})$
- Exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ (in all dimensions)

$$\boxed{\Lambda^m = \Lambda^m_+ \oplus \Lambda^m_- \text{ in dimension } 2m}$$

- Conformal Killing $X^a \mapsto \nabla_{(a}X_{b)} -$ trace
- Dirac operator
- Rarita-Schwinger operator
- Yamabe operator (aka conformal Laplacian, $\Delta + \cdots$)
- Paneitz operator $(\Delta^2 + \cdots)$, etc etc etc

Meaning?

WARNING! $\exists \geq 8$ possible formulations! But...

- Invariance as operator on conformal manifold
- Invariance on S^n under action of $\mathrm{SO}(n+1,1)$ or Spin

Naïve verification $g_{ab} \rightsquigarrow \hat{g}_{ab} = \Omega^2 g_{ab} \;\Rightarrow\; \nabla_a \rightsquigarrow \hat{\nabla}_a$

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b - \Upsilon^b X_a + \Upsilon_c X^c \delta_a{}^b$$

where $\Upsilon_a \equiv (\nabla_a \Omega)/\Omega$.

$$\hat{\nabla}_a X_b = \nabla_a X_b + \Upsilon_a X_b - \Upsilon_b X_a + \Upsilon_c X^c g_{ab}$$

Hence $\quad \hat{\nabla}_{(a} X_{b)} = \nabla_{(a} X_{b)} + \Upsilon_c X^c g_{ab}$

Conformal Killing $\quad \checkmark$

Yamabe operator

Conformal densities of weight w $\quad f \rightsquigarrow \hat{f} = \Omega^w f$

$$\hat\nabla_a \omega_b = \nabla_a \omega_b + (w-1)\Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^c \omega_c g_{ab}$$

$$\begin{aligned} \hat\nabla_a \hat\nabla_b f &= \hat\nabla_a(\nabla_b f + w\Upsilon_b f) \\ &= \nabla_a(\nabla_b f + w\Upsilon_b f) \\ &\quad + (w-1)\Upsilon_a(\nabla_b f + w\Upsilon_b f) \\ &\quad - \Upsilon_b(\nabla_a f + w\Upsilon_a f) \\ &\quad + \Upsilon^c(\nabla_c f + w\Upsilon_c f) g_{ab} \end{aligned}$$

$$\therefore \quad \boxed{\begin{aligned} \hat\Delta f &= \Delta f + (n+2w-2)\Upsilon^a \nabla_a f \\ &\quad + w(\nabla^a \Upsilon_a + (n+w-2)\Upsilon^a \Upsilon_a) f \end{aligned}}$$

Yamabe operator cont'd

$$\begin{aligned}\hat{\Delta} f &= \Delta f + (n + 2w - 2)\Upsilon^a \nabla_a f \\ &\quad + w(\nabla^a \Upsilon_a + (n + w - 2)\Upsilon^a \Upsilon_a) f\end{aligned}$$

If $w = 1 - n/2$

$$\hat{\Delta} f = \Delta f - \tfrac{1}{4}(n-2)(2\nabla^a \Upsilon_a + (n-2)\Upsilon^a \Upsilon_a) f$$

But $\hat{R} = R - (n-1)(2\nabla^a \Upsilon_a + (n-2)\Upsilon^a \Upsilon_a)$

$$\therefore \quad \hat{\Delta} - \tfrac{n-2}{4(n-1)}\hat{R} = \Delta - \tfrac{n-2}{4(n-1)} R$$ invariant

$$L \equiv \Delta - \tfrac{n-2}{4(n-1)} R : \Lambda^0[1 - n/2] \to \Lambda^0[-1 - n/2]$$

SO(n+1,1)-invariance

Recall $S^n = \mathrm{SO}(n+1,1)/P = G/P$, where

$$\mathfrak{p} = \underbrace{\text{rotations} \oplus \text{dilations}}_{} \oplus \text{inversions}$$

Levi subalgebra $\rightsquigarrow \mathrm{SO}(n) \times \{\lambda > 0\}$

Irreducible homogeneous vector bundles on S^n

$\longleftrightarrow$ irreducible $\mathrm{SO}(n)$-module $\otimes$ $\boxed{\lambda \mapsto \lambda^{-w}}$

$\longleftrightarrow V[w]$

irreducible Riemannian tensor bundle → V

conformal weight w → w

EG: Λ^p, $\bigodot_\circ^b \Lambda^1$, $\boxminus$, $\boxplus_\circ$, $\boxplus_\circ$, …
(or $\pm$-part thereof)

Operators on the three-sphere

A complete list of SO(4, 1)-invariant linear differential operators between irreducible tensor bundles

- Standard (with suitable conformal weights)

$$\bigodot_\circ^b \Lambda^1 \xrightarrow{\nabla^{a+1}} \bigodot_\circ^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{2b+1}} \bigodot_\circ^{a+b+1} \Lambda^1 \xrightarrow{\nabla^{a+1}} \bigodot_\circ^b \Lambda^1$$

for $a, b \in \mathbb{Z}_{\geq 0}$ ($a = b = 0 \rightsquigarrow$ de Rham complex)

- Non-standard

$$\bigodot_\circ^b \Lambda^1[a + 2b] \xrightarrow{\nabla^{2a+2b+3}} \bigodot_\circ^b \Lambda^1[-a - 3]$$

for $a + 1/2, b \in \mathbb{Z}_{\geq 0}$ ($a = -1/2, b = 0 \rightsquigarrow$ Laplacian)

Proof is by algebra (Lie theory and Verma modules)

Theorem All these operators have conformally invariant 'curved analogues.'

Conformal-Einstein operator

Let $\mathrm{P}_{ab} \equiv \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}Rg_{ab}\right)$. Then

$$\sigma \stackrel{D}{\longmapsto} \text{trace-free part of}(\nabla_a\nabla_b\sigma + \mathrm{P}_{ab}\sigma)$$

is conformally invariant, σ weight 1 $(a = 1, b = 0)$

Geometric meaning where $\sigma \neq 0$ (LeBrun 1985)

$$D\sigma = 0 \iff \sigma^{-2}g_{ab} \text{ is an Einstein metric}$$

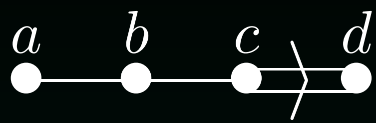
Prolong

$$D\sigma = 0 \iff \begin{aligned} \nabla_a\sigma - \mu_a &= 0 \\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho &= 0 \\ \nabla_a\rho - \mathrm{P}_a{}^b\mu_b &= 0 \end{aligned}$$

Cartan connection

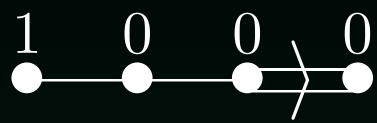
$\rightsquigarrow$ Curved translation principle

Representation theory

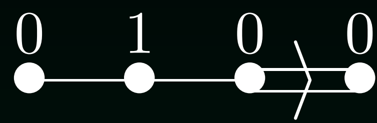
SO(9) 

$a, b, c \in \mathbb{Z}_{\geq 0}, \quad d \in 2\mathbb{Z}_{\geq 0}$

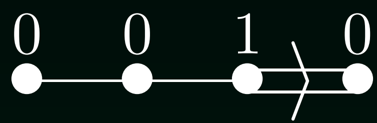
EG:



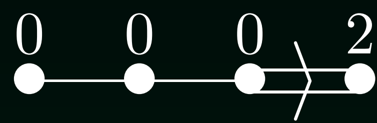
$= \mathbb{R}^9$



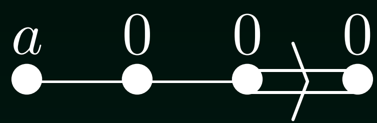
$= \Lambda^2\mathbb{R}^9$



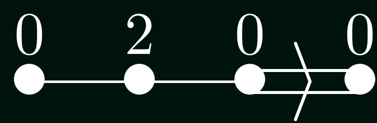
$= \Lambda^3\mathbb{R}^9$



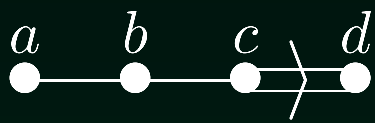
$= \Lambda^4\mathbb{R}^9$



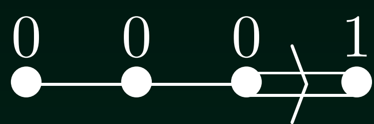
$= \bigodot^a_\circ \mathbb{R}^9$



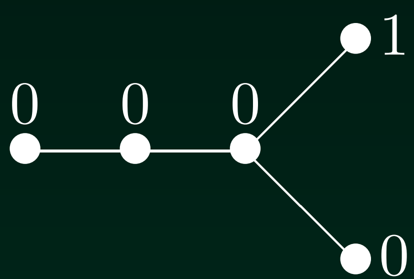
$= \boxplus_\circ$

Spin(9) 

$a, b, c, d \in \mathbb{Z}_{\geq 0}$

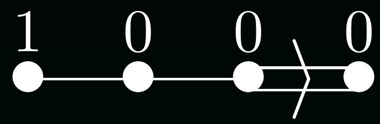
EG: 

$= \mathbb{S}$, basic spin representation

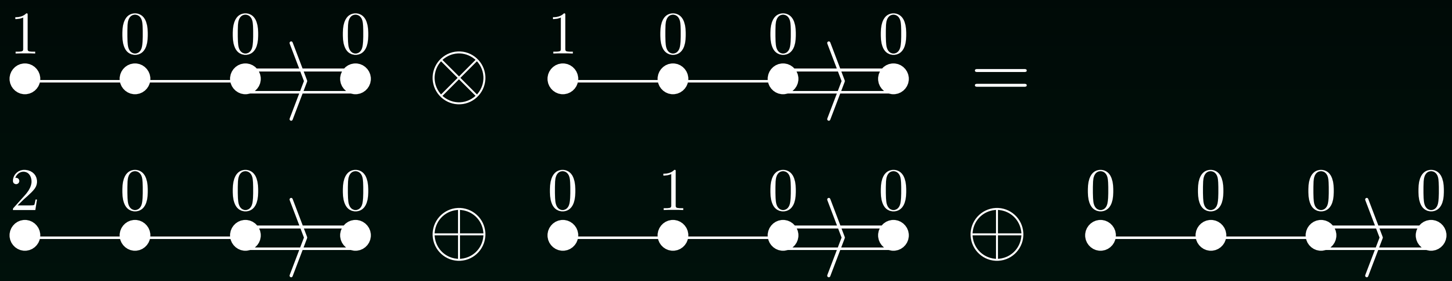
Spin(10) 

$= \mathbb{S}^+$

Representation theory cont'd

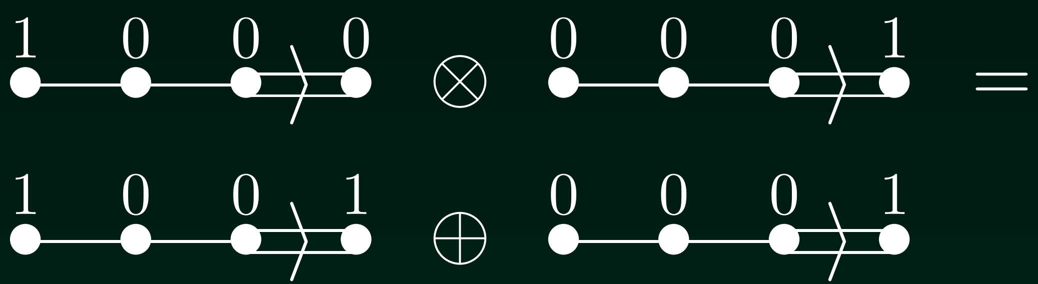
On a Riemannian 9-manifold 

$\leftrightsquigarrow \Lambda^1$



$\nabla_{(a}\omega_{b)} - \text{trace}$ $\quad$ $\nabla_{[a}\omega_{b]}$ $\quad$ $\nabla^a\omega_a$

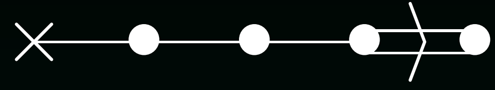
On a 9-dimensional spin manifold



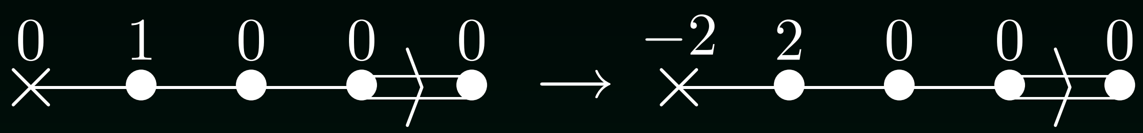
twistor operator $\quad$ Dirac operator

Some invariant operators

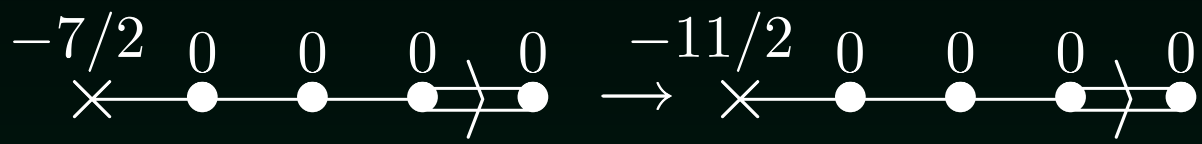
$S^9 = \mathrm{Spin}(10,1)/P$ =



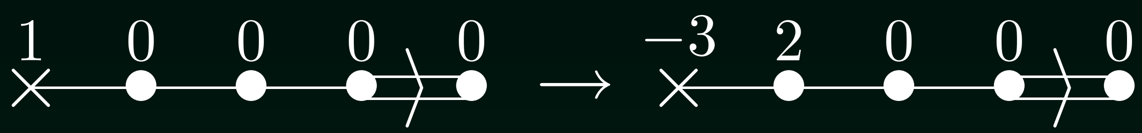
Killing



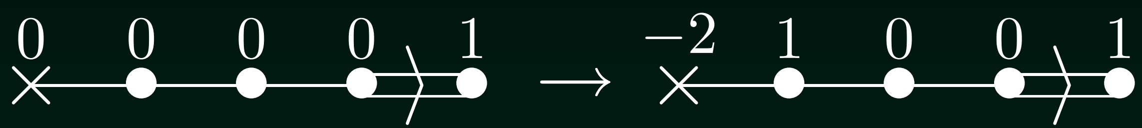
Yamabe



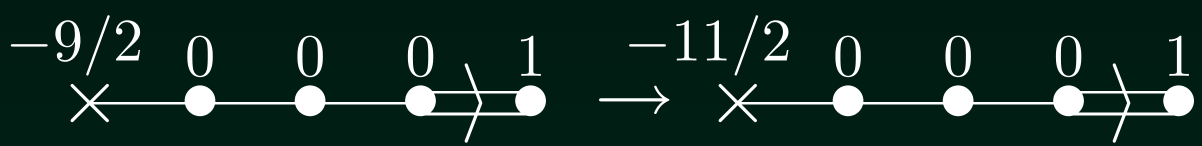
Conformal-Einstein



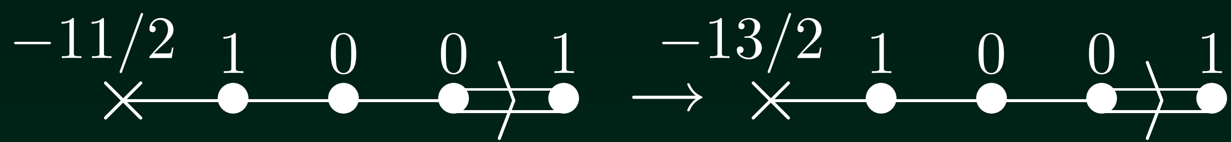
Twistor



Dirac

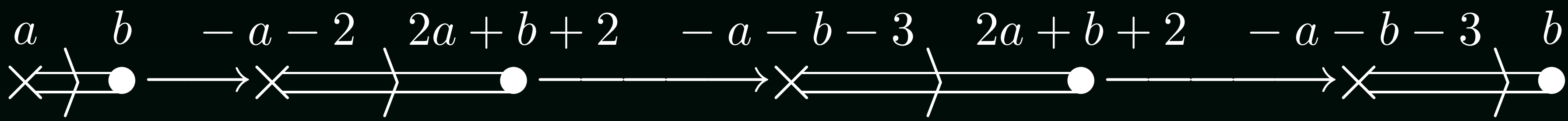


Rarita-Schwinger

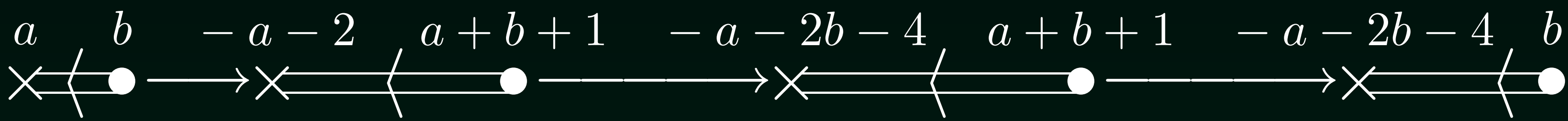


BGG complexes on 3-sphere

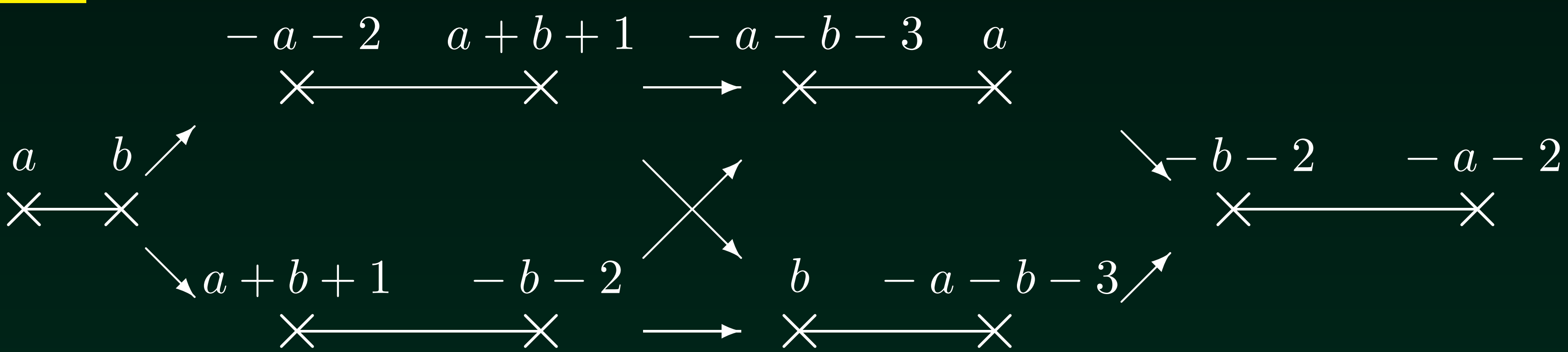
Conformal



Contact projective

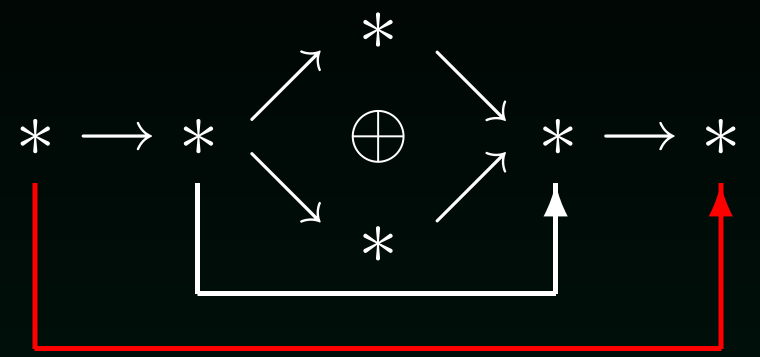


CR



Beware the four-sphere

Pattern



cf. de Rham
Maxwell

Theorem Most of these operators have conformally invariant ‘curved analogues.’

Standard ✓ Non-standard ☹

Δ	✓	Bateman, Yamabe, et alia
Δ^2	✓	Paneitz, et alia, GJMS
Δ^3	✗	Graham
$\Delta^{\geq 4}$	✗	Gover & Hirachi
general	☹	cf. Eastwood & Slovák

References

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THANK YOU

END OF PART TWO