

**Conformal differential geometry
and its interaction with representation theory**

Higher symmetries of the Laplacian

Michael Eastwood

Australian National University

References

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- ME and Thomas Leistner, *Higher symmetries of the square of the Laplacian*, IMA Volumes 144, Springer Verlag 2007, pp. 319–338.

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A simple question on \mathbb{R}^n , $n \geq 3$

Question: Which linear differential operators preserve harmonic functions? **Answer on \mathbb{R}^3 :**–

Zeroth order $f \mapsto \text{constant} \times f$ 1

First order

$$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3 \quad \text{3}$$

$$x_1 \nabla_2 - x_2 \nabla_1 \quad \&c. \quad \text{3}$$

$$x_1 \nabla_1 + x_2 \nabla_2 + x_3 \nabla_3 \quad \boxed{+1/2} \quad \text{1}$$

$$(x_1^2 - x_2^2 - x_3^2) \nabla_1 + 2x_1 x_2 \nabla_2 + 2x_1 x_3 \nabla_3 + x_1 \quad \text{3}$$

&c. =====

Dimensions 10

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1$$

Lie Algebra = $\mathfrak{so}(4, 1)$ = conformal algebra ← NB!

Second order

Boyer-Kalnins-Miller (1976)

Extras: \propto Laplacian ($f \mapsto h\Delta f$ for any smooth h)
plus a 35-dim^ℓ family of new ones!

$$\{\mathcal{D}_1, \mathcal{D}_2\} \equiv \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1$$

$$\odot^2 \mathfrak{so}(4, 1) = ? \quad \dim = 10 \times 11/2 = 55$$

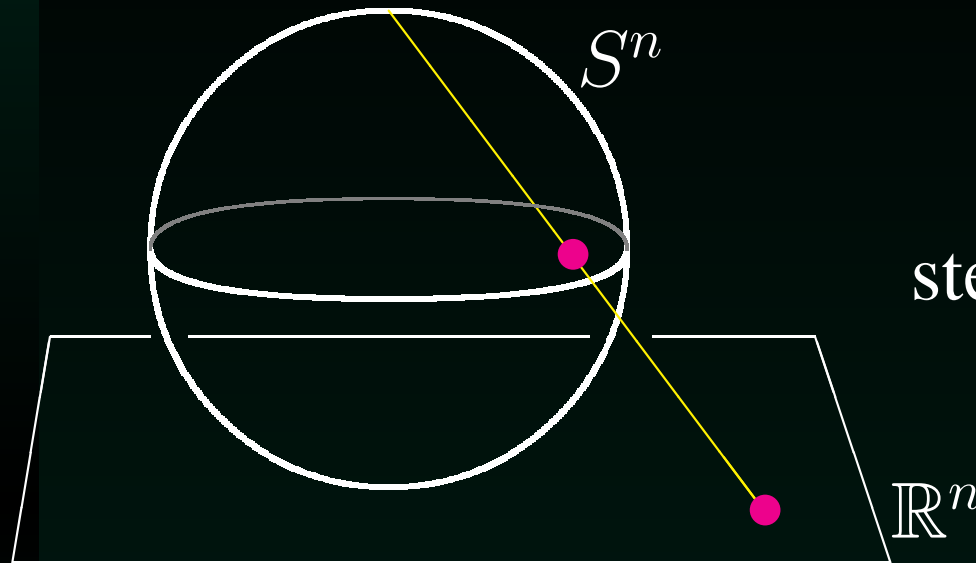
$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \odot \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \oplus \mathbb{R} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

$$55 = 35 + 14 + 1 + 5$$

Separation of variables (Bôcher, Bateman, ...).

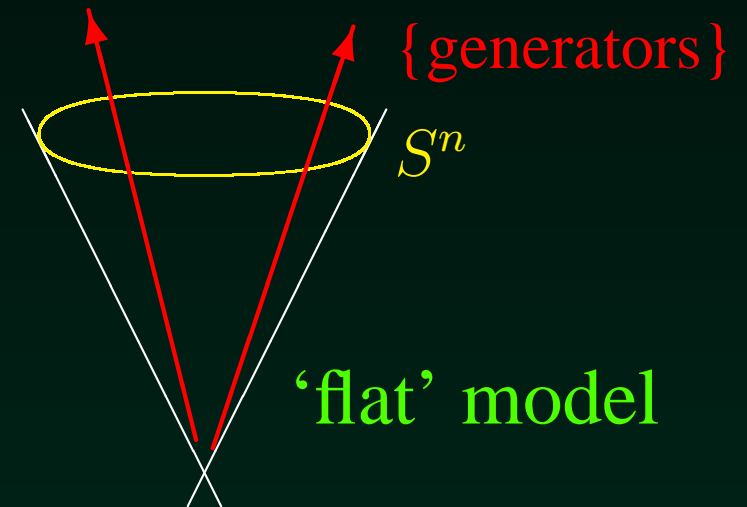
Third order...?

Conformal geometry



stereographic projection

Action of $SO(n + 1, 1)$ on S^n
by conformal transformations



Conformal Laplacian Dirac 1935

$$r \equiv x_1^2 + \cdots + x_n^2 + x_{n+1}^2 - x_{n+2}^2$$

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2} - \frac{\partial^2}{\partial x_{n+2}^2}$$

f on null cone $\subset \mathbb{R}^{n+2}$ homogeneous of degree $w \rightsquigarrow$

- ambiently extend to \tilde{f} of degree w
- freedom $\tilde{f} \mapsto \tilde{f} + rg$ for g of degree $w - 2$
- calculate: $\tilde{\Delta}(rg) = r\tilde{\Delta}g + 2(n + 2w - 2)g$

$w = 1 - n/2 \Rightarrow f \mapsto (\tilde{\Delta}\tilde{f})|_{r=0}$ is invariantly defined.

On \mathbb{R}^n it's Δ

On S^n it's $\Delta - \frac{n-2}{4(n-1)}R$

AdS/CFT

Fefferman-Graham 'ambient' metric

Symmetries of Δ

\mathcal{D} a symmetry $\iff \Delta\mathcal{D} = \delta\Delta$ for some δ .

trivial example: $\mathcal{D} = \mathcal{P}\Delta$ for any \mathcal{P}

equivalence: $\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$

$\mathbb{R}^n \rightsquigarrow \mathcal{A}_n \equiv$ algebra of symmetries

under composition
up to equivalence

Write $\mathcal{D} = \underline{V^{bc\dots d}} \nabla_b \nabla_c \dots \nabla_d +$ lower order terms

symbol

normalise w.l.g. to be **trace-free**

Theorems

→ \mathcal{D} a symmetry \Rightarrow trace-free part of $\nabla^{(a} V^{bc \cdots d)} = 0$

→ Easy

→ On \mathbb{R}^n , such a conformal Killing tensor $V^{bc \cdots d} \rightsquigarrow \mathcal{D}_V$

→ Not So Easy

\mathcal{D}_V is a canonically associated symmetry of the form

$$\mathcal{D}_V = V^{bc \cdots d} \nabla_b \nabla_c \cdots \nabla_d + \text{lower order terms.}$$

- E.g. First order

$$\mathcal{D}_V f = V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

- E.g. Second order

$$\mathcal{D}_V f = V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f$$

Ingredients of proof

- We can solve the conformal Killing tensor equation

$$\nabla^{(a} V^{bc\dots d)} = g^{(ab} \lambda^{c\dots d)}$$

on \mathbb{R}^n by prolongation and/or BGG machinery:-

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|} \hline & & & \dots & & & \\ \hline & & & \dots & & & \\ \hline \end{array}}_{\circ} \quad \text{w.r.t. } \mathfrak{so}(n+1, 1).$$

of columns = # of indices on $V^{bc\dots d}$

$$\begin{aligned} \text{E.g. } V^b &= s^b + m^{bc} x_c + \lambda x^b + r^c x_c x^b - \frac{1}{2} x^c x_c r^b \\ &= \text{translation} + \text{rotation} + \text{dilation} + \text{inversion}. \end{aligned}$$

- Use ‘ambient’ methods to construct \mathcal{D}_V .

Corollary

As a vector space

$$\mathcal{A}_n = \bigoplus_{s=0}^{\infty} \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & \cdots & & & & \\ \hline & & & \cdots & & & & \\ \hline \end{array}}_s \circ$$

Question: What about the algebra structure?

Cf.: let \mathfrak{g} be a Lie algebra. As a vector space

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{s=0}^{\infty} \odot^s \mathfrak{g}$$

but the algebra structure is opaque viewed this way.

The algebra structure

$$\begin{aligned} \mathfrak{U}(\mathfrak{g}) &= \frac{\otimes \mathfrak{g}}{(X \otimes Y - Y \otimes X - [X, Y])} \\ &= \frac{\otimes \mathfrak{g}}{(X \otimes Y - X \odot Y - \frac{1}{2}[X, Y])} \end{aligned}$$

Theorem

$$\mathcal{A}_n = \frac{\otimes \mathfrak{so}(n+1, 1)}{\underbrace{(X \otimes Y - X \odot Y - \frac{1}{2}[X, Y])}_{\text{Cartan}} + \underbrace{\frac{n-2}{4n(n+1)} \langle X, Y \rangle}_{\text{Lie}} \underbrace{\langle X, Y \rangle}_{\text{Killing}}}$$

Equivalently,

$$\mathcal{A}_n = \mathfrak{U}(\mathfrak{so}(n+1, 1)) / \underline{\text{Joseph Ideal}}.$$

Proof of algebra structure

Calculate by ambient means that

$$\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \frac{1}{2} \mathcal{D}_{[X, Y]} - \frac{n-2}{4n(n+1)} \mathcal{D}_{\langle X, Y \rangle}$$

and use properties of Cartan product (due to Kostant).

Remark: simple Lie algebra $\neq \mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C}) \Rightarrow$

$$\dim \frac{\bigotimes \mathfrak{g}}{(X \otimes Y - X \odot Y - \frac{1}{2}[X, Y] - \lambda \langle X, Y \rangle)} = \infty$$

for precisely one value of λ (Braverman and Joseph)

$$\rightsquigarrow \text{graded algebra } \bigoplus_{s=0}^{\infty} \odot^s \mathfrak{g}.$$

Curved analogues

For any vector field V^a ,

$$f \mapsto V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

is conformally invariant.

For any trace-free symmetric tensor field V^{ab} ,

$$f \mapsto V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f$$

$$- \frac{n+2}{4(n+1)} R_{ab} V^{ab} f =$$

curvature
correction
terms

is conformally invariant &c. &c.

Curved symmetries?

- V^a is a conformal Killing vector \Rightarrow

$$\mathcal{D}_V f \equiv V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

is symmetry of the conformal Laplacian.

- V^{ab} is a conformal Killing tensor $\Rightarrow?$

$$\begin{aligned} \mathcal{D}_V f &\equiv V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f \\ &+ \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f - \frac{n+2}{4(n+1)} R_{ab} V^{ab} f \end{aligned}$$

is a symmetry of the conformal Laplacian. **Unknown!**

Another operator

In even dimensions, there is the Dirac operator

$$D : \mathbb{S}^+ \rightarrow \mathbb{S}^-.$$

A symmetry of D is an operator $\mathcal{D} : \mathbb{S}^+ \rightarrow \mathbb{S}^+$ s.t.

$$\begin{array}{ccc} \mathbb{S}^+ & \xrightarrow{D} & \mathbb{S}^- \\ \mathcal{D} \downarrow & & \delta \downarrow \\ \mathbb{S}^+ & \xrightarrow{D} & \mathbb{S}^- \end{array}$$

commutes for some differential operator $\delta : \mathbb{S}^- \rightarrow \mathbb{S}^-$.

- The symbol of \mathcal{D} satisfies a conformally invariant overdetermined system of equations.
- ✓ First order symmetries: Benn and Kress.
- ✓ Higher order symmetries in the flat case: E, Somberg, and Souček.

Yet another operator

For the square of the Laplacian (E and Leistner)
symmetry algebra =

$$\otimes \mathfrak{so}(n+1, 1)$$

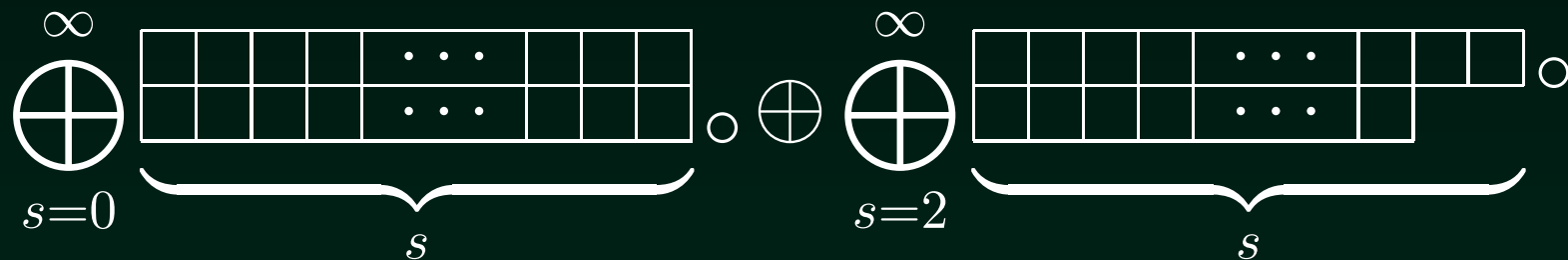
$$\left(X \otimes Y - X \odot Y - X \bullet Y - \frac{1}{2}[X, Y] + \frac{(n-4)(n+4)}{4n(n+1)(n+2)} \langle X, Y \rangle \right)$$

and some fourth order elements

new

different

with graded counterpart



THANK YOU

END OF PART THREE