

**Conformal differential geometry
and its interaction with representation theory**

Twistor theory and the harmonic hull

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Harmonic functions

The Laplacian on \mathbb{R}^n : $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$

$\Delta u = 0 \implies u$ is real-analytic

$$\therefore u(x_1, x_2, \dots, x_n) \rightsquigarrow \widehat{u}(z_1, z_2, \dots, z_n)$$

More precisely,

$$\begin{array}{ccccc} \mathbb{R}^n & \supseteq_{\text{open}} & U & \xrightarrow{u} & \mathbb{R} \text{ or } \mathbb{C} \\ & \cap & \cap & & \cap \\ \mathbb{C}^n & \supseteq_{\text{open}} & \widehat{U} & \xrightarrow{\widehat{u}} & \mathbb{C} \end{array}$$

FAQ:

Depends on u ?

How big?

2 dimensions

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \underbrace{\left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)}_{\text{Cauchy-Riemann}}$$

- $\mathbb{R}^2 \supseteq_{\text{open}} U \xrightarrow{u} \mathbb{C}$ harmonic
 - U simply-connected
- $$\left. \vphantom{\begin{matrix} \bullet \\ \bullet \end{matrix}} \right\} \implies$$

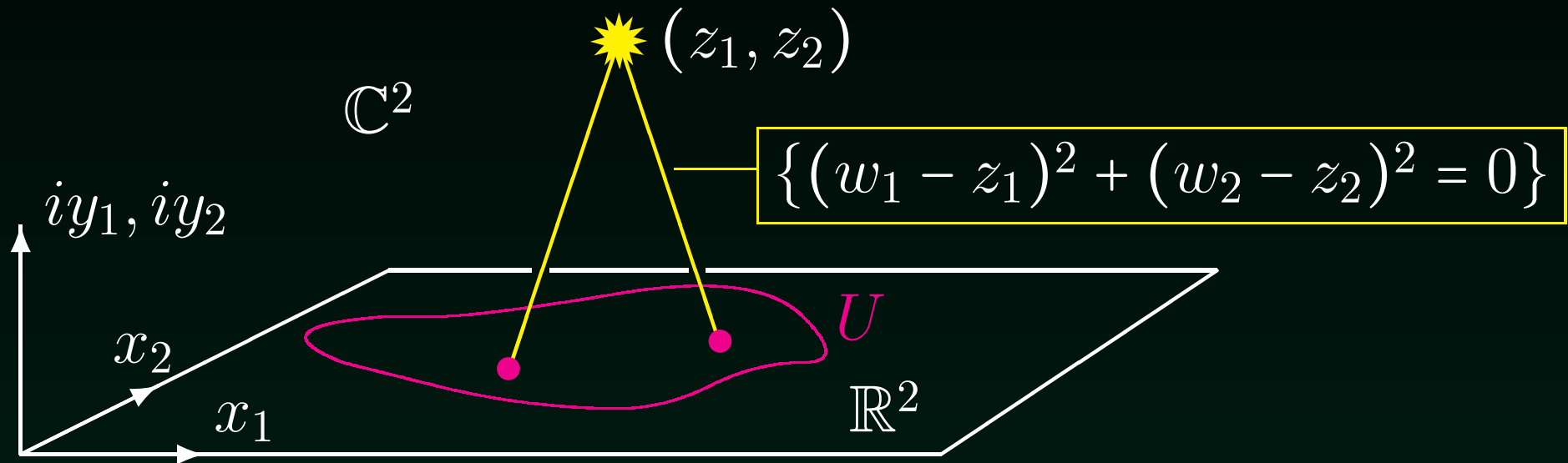
$$u(x_1, x_2) = \underbrace{f(x_1 + ix_2)}_{\text{holomorphic}} + \underbrace{g(x_1 - ix_2)}_{\text{holomorphic}}$$

$$\therefore \widehat{u}(z_1, z_2) = f(z_1 + iz_2) + g(z_1 - iz_2)$$

where it makes sense: $z_1 + iz_2 \in U$ and $z_1 - iz_2 \in \overline{U}$

2 dimensions cont'd

$$\boxed{z_1 + iz_2 \in U} \text{ and } \boxed{z_1 - iz_2 \in \overline{U}} \iff$$



\therefore all u harmonic on U extend to \tilde{u} holomorphic on

$$\tilde{U} \equiv \{z \in \mathbb{C}^2 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^2 \subset U\}$$

where $\mathcal{N}(z) = \{w \in \mathbb{C}^2 \text{ s.t. } (w - z)^2 = 0\}$ Null 'Cone'

2 dimensions concl'd

If $U \subseteq \mathbb{R}^2$ is simply-connected then

- all harmonic functions on U extend to $\tilde{U} \subseteq \mathbb{C}^2$,
- \tilde{U} is connected (and simply-connected),
- \tilde{U} is maximal in this respect \star .

\tilde{U} is called the harmonic hull of U .

\star consider $\log(z - x)^2$ for $x \in \partial U$.

If $U \subset \mathbb{R}^2$ is multiply-connected, it does not have a harmonic hull: consider

$\log(z - x)^2$ for $x \in \mathbb{R}^2 \setminus U$ surrounded by U .

4 dimensions

Bateman

$$u(x) = \oint_{\gamma} f((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta) d\zeta$$

a holomorphic function of 3 variables
a closed contour in the complex plane

Differentiation under the integral sign \implies

$$\Delta u = 0$$

John: $u(w, x, y, z) = \int_{-\infty}^{\infty} f(w + xt, y + zt, t) dt \implies$

$$u_{wz} - u_{xy} = 0$$

and... X-ray

FAQ (circa 1904–1980)

- where is f defined?
- where is γ located?
- which u arise in this way?

answer by means of
Penrose transform

Suspend disbelief!

4 dimensions cont'd

Naive use of Bateman

$$u(x) = \oint_{\gamma} f \left((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta \right) d\zeta$$

$$u(z) = \oint_{\gamma} f \left((z_1 + iz_2) + (iz_3 + z_4)\zeta, (iz_3 - z_4) + (z_1 - iz_2)\zeta, \zeta \right) d\zeta$$

Perhaps we should insist that

$f(z, \zeta)$ be defined wherever $f(x, \zeta)$ is defined i.e.

for fixed $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$,

$$L_z \equiv \left\{ \left((z_1 + iz_2) + (iz_3 + z_4)\zeta, (iz_3 - z_4) + (z_1 - iz_2)\zeta, \zeta \right) \text{ s.t. } \zeta \in \mathbb{C} \right\}$$

\cap WANT! \swarrow open in \mathbb{C}^3

$$\left\{ \left((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta \right) \text{ s.t. } x \in U, \zeta \in \mathbb{C} \right\}$$

What we want

Fix $z \in \mathbb{C}^4$. For any $\zeta \in \mathbb{C}$, we want to be able to solve

$$\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix} = \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix}$$

for some $x = (x_1, x_2, x_3, x_4) \in U \subseteq^{\text{open}} \mathbb{R}^4$. Necessarily,

$$\det \left(\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix} - \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix} \right) = 0$$

i.e. $(x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2 + (x_4 - z_4)^2 = 0$

(also if $\zeta = \infty$). This is also sufficient: we require

$$\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \quad \text{where } \mathcal{N}(z) = \text{null cone based at } z$$

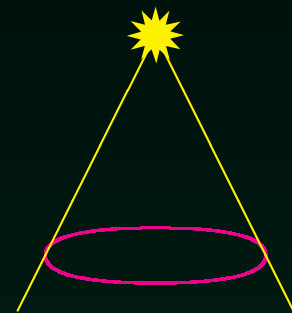
4 dimensions concl'd

For any U open, connected $\subseteq \mathbb{R}^4$, let

$$\tilde{U} \equiv \{z \in \mathbb{C}^4 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^4 \subset U\}^{\text{connected}}$$

where $\mathcal{N}(z) = \{w \in \mathbb{C}^4 \text{ s.t. } (w - z)^2 = 0\}$ Null Cone

- All harmonic functions on U extend to \tilde{U} ,
- \tilde{U} is maximal \star (harmonic hull).



\star Consider $1/(z - x)^2$ for $x \in \partial U$.

NB $n = 4$ is better than $n = 2$ if Bateman OK.
??????????????

The twistor fibration

Looking at Bateman's formula, consider

$$\begin{array}{ccc} \zeta \in \mathbb{C} \hookrightarrow \mathbb{C}^3 & \text{write } L_x \text{ for the range, where } x \in \mathbb{R}^4 & \\ \downarrow & \Psi & \\ ((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta) & & \end{array}$$

Compactify! $\mathbb{C}\mathbb{P}_1 \hookrightarrow \mathbb{C}\mathbb{P}_3$ (still write range as L_x)

Foliation of $\mathbb{C}\mathbb{P}_3 \setminus \{[* , * , 0 , 0]\} \hookrightarrow \mathbb{C}\mathbb{P}_3$

$$\begin{array}{ccc} & \downarrow & \downarrow \tau \text{ submersion} \\ L_x = \tau^{-1}(x) & \mathbb{R}^4 & S^4 \\ & \begin{array}{c} \hookrightarrow \\ \longleftarrow \\ \text{stereographic projection} \end{array} & \end{array}$$

cf. Hopf

The Penrose transform

Theorem For any $U^{\text{open}} \subseteq \mathbb{R}^4 \hookrightarrow S^4 \xleftarrow{\tau} \mathbb{C}\mathbb{P}_3$

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\cong} \{u : U \rightarrow \mathbb{C} \text{ s.t. } \Delta u = 0\}.$$

Interprets Bateman's formula and answers FAQ!

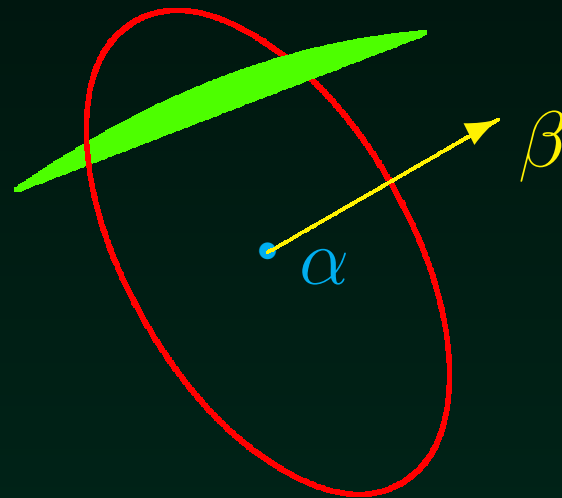
- $\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \iff L_z \subset \tau^{-1}(U)$
- \tilde{U} = harmonic hull, as suspected
- For $U^{\text{open}} \subseteq S^4$, $\Delta \rightsquigarrow \Delta - \frac{1}{6}R$ (Yamabe)
- OK in higher even dimensions (M.K. Murray).

Geometry of the harmonic hull

For $z = \alpha + i\beta \in \mathbb{C}^n$

$$\begin{aligned}\mathcal{N}(z) \cap \mathbb{R}^n &= \{x \text{ s.t. } (x - z)^2 = 0\} \\ &= \{x \text{ s.t. } |x - \alpha|^2 = |\beta|^2 \ \& \ (x - \alpha) \cdot \beta = 0\}\end{aligned}$$

NB $\mathcal{N}(z) \cap \mathbb{R}^n = \mathcal{N}(\bar{z}) \cap \mathbb{R}^n = \underline{\text{Round } (n - 2)\text{-sphere}}$



Odd dimensions

$U^{\text{open, connected}} \subseteq \mathbb{R}^{\text{odd}}$ need not have a harmonic hull!

Example $\mathbb{R}^3 \setminus \{0\}$ Reasons

Recall $\mathbb{C}^3 \ni z \rightsquigarrow \mathcal{N}(z) \cap \mathbb{R}^3 \equiv \text{Circle}(z) \rightsquigarrow \text{Disc}(z)$

- OK extension to $\{z \text{ s.t. } \text{Disc}(z) \neq 0\}$
Aronszajn, Creese, and Lipkin
- $f(x) \rightsquigarrow \frac{1}{\sqrt{1-2r \cdot x + |r|^2|x|^2}} f\left(\frac{x-r|x|^2}{1-2r \cdot x + |r|^2|x|^2}\right)$ harmonic ✓
- Hence should extend to $\mathbb{C}^3 \setminus \{z_1^2 + z_2^2 + z_3^2 = 0\}$
- But $1/\sqrt{z_1^2 + z_2^2 + z_3^2}$ branches!!

Cf. Huygens, Kirchhoff, et cetera

Further Reading

- N. Aronszajn, T. Creese, and L. Lipkin, *Polyharmonic Functions*, Oxford University Press 1983.
- H. Bateman, The solution of partial differential equations by means of definite integrals, *Proc. Lond. Math. Soc.* **1** (1904) 451–458.
- M.G. Eastwood and F. Xu, Twistor theory and the harmonic hull, [arXiv:1012.2620](https://arxiv.org/abs/1012.2620).
- B. Kroetz and H. Schlichtkrull, Holomorphic extension of eigenfunctions, [arXiv:0812.0724](https://arxiv.org/abs/0812.0724).
- M.K. Murray, A Penrose transform for the twistor space of an even-dimensional conformally flat Riemannian manifold, *Ann. Global Anal. Geom.* **4** (1986) 71–88.

THANK YOU

END OF PART FOUR