

Conformal foliations and CR geometry

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[joint work with Paul Baird]

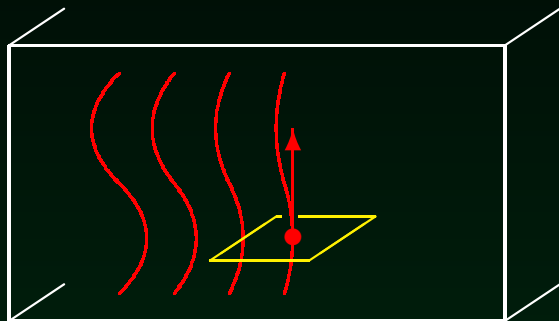
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Disclaimers and references

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Conformal foliations

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.



U is (transversally) conformal
 $\Leftrightarrow \mathcal{L}_U$ preserves the conformal metric orthogonal to its leaves

h
 \mathbb{C}

isothermal
coördinates



$$h = f + ig \quad \langle \nabla f, \nabla g \rangle = 0$$
$$\|\nabla f\| = \|\nabla g\|$$

conjugate functions

Conjugate functions on \mathbb{R}^2

$$f = f(r, s) \quad g = g(r, s) \quad \text{s.t.} \quad \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $f = r \quad g = s$
- $f = r^2 - s^2 \quad g = 2rs$
- $f = \frac{r}{r^2 + s^2} \quad g = \frac{s}{r^2 + s^2}$
- $f = e^r \cos s \quad g = e^r \sin s$

$h \equiv f + ig$ is (anti-)holomorphic in $z \equiv r + is$

Conjugate functions on \mathbb{R}^3

$$f = f(q, r, s) \quad g = g(q, r, s) \quad \text{s.t.} \quad \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $f = r \quad g = s$

- $f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2 + s^2}$

- $f = r \frac{q^2 + r^2 + s^2}{r^2 + s^2} \quad g = s \frac{q^2 + r^2 + s^2}{r^2 + s^2}$

- $$f = \frac{(1 - q^2 - r^2 - s^2)r + 2qs}{r^2 + s^2}$$

$$g = \frac{(1 - q^2 - r^2 - s^2)s - 2qr}{r^2 + s^2}$$

$$\begin{array}{ccc} \mathbb{R}^3 & \hookrightarrow & S^3 \\ & & \downarrow \text{Hopf} \\ \mathbb{R}^2 & \leftarrow & S^2 \setminus \{*\} \end{array}$$

Almost Hermitian structures

NB: $J(p, q, r, s) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ satisfies

- $J^2 = -\text{Id}$
 - $J \in \text{SO}(4)$
- $$\iff J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & 0 & -u \\ w & -v & u & 0 \end{bmatrix}$$
- $u^2 + v^2 + w^2 = 1$, two-sphere

Consider $\mathbb{R}^3 = \{(p, q, r, s) \in \mathbb{R}^4 \mid p = 0\} \subset \mathbb{R}^4$

NB: $U \equiv \left(J \frac{\partial}{\partial p} \right) \Big|_{\mathbb{R}^3} = \left(u \frac{\partial}{\partial q} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial s} \right) \Big|_{\mathbb{R}^3}$

unit vector field

also \leadsto two-sphere

Sphere bundles

bundle of
unit vectors

bundle of almost
Hermitian structures

$$\begin{array}{ccc}
 Q_0 & \subset & Z_0 \\
 \downarrow \tau & & \downarrow \tau \\
 \mathbb{R}^3 & \subset & \mathbb{R}^4
 \end{array}$$

section



unit vector field

section



almost Hermitian structure

Hermitian structures

Lemma

J is integrable $\implies U \equiv \left(J \frac{\partial}{\partial p} \right) \Big|_{\mathbb{R}^3}$ is conformal

Conversely??

NB: J integrable $\implies J$ real-analytic

Question: U conformal $\implies U$ real-analytic??

Answer: NO!

However: U real-analytic and conformal
 $\implies U$ extends uniquely to an integrable J .

} WHY?

Twistor geometry

bundle of almost
Hermitian structures

$$\begin{array}{ccc}
 Q_o \subset Z_o & \cong & \mathbb{C}P_3 \setminus \{z_3 = z_4 = 0\} \ni [z_1, z_2, z_3, z_4] \\
 \downarrow & \tau \downarrow & \downarrow \\
 \mathbb{R}^3 \subset \mathbb{R}^4 & \cong & \mathbb{C}^2 \ni \begin{pmatrix} p + iq \\ r + is \end{pmatrix} = \frac{1}{|z_3|^2 + |z_4|^2} \begin{pmatrix} z_2 \bar{z}_3 + z_4 \bar{z}_1 \\ z_1 \bar{z}_3 - z_4 \bar{z}_2 \end{pmatrix}
 \end{array}$$

compactify

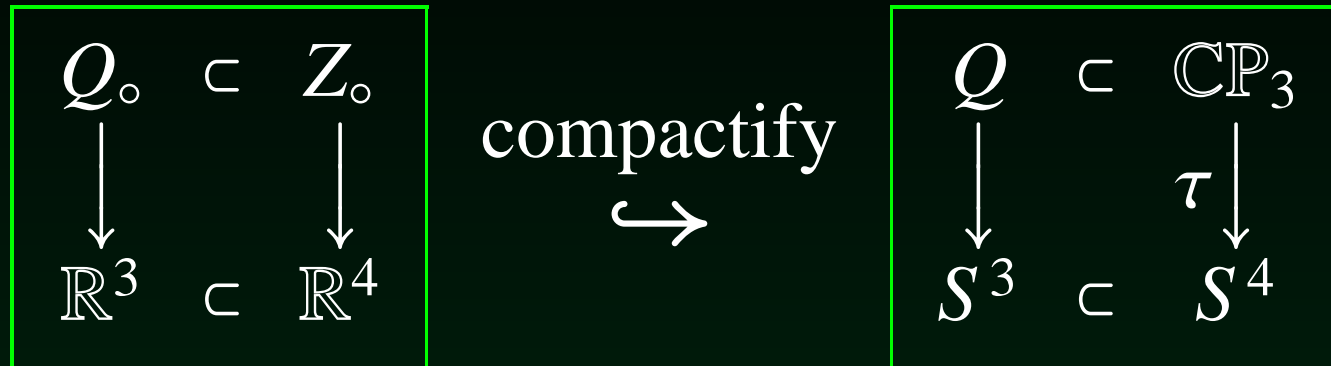
$\mathbb{C}P_3$

$\tau \downarrow$

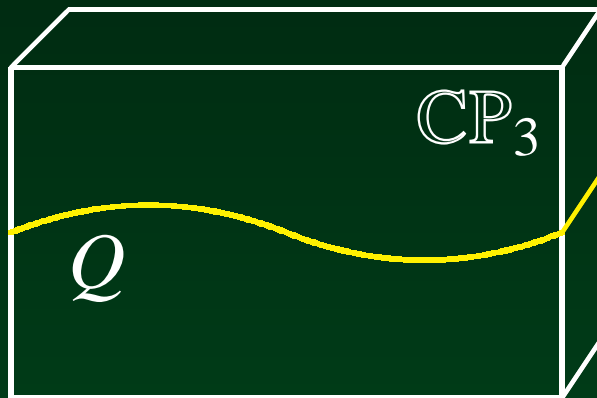
S^4

twistor fibration (cf. Hopf)

Twistor geometry cont'd



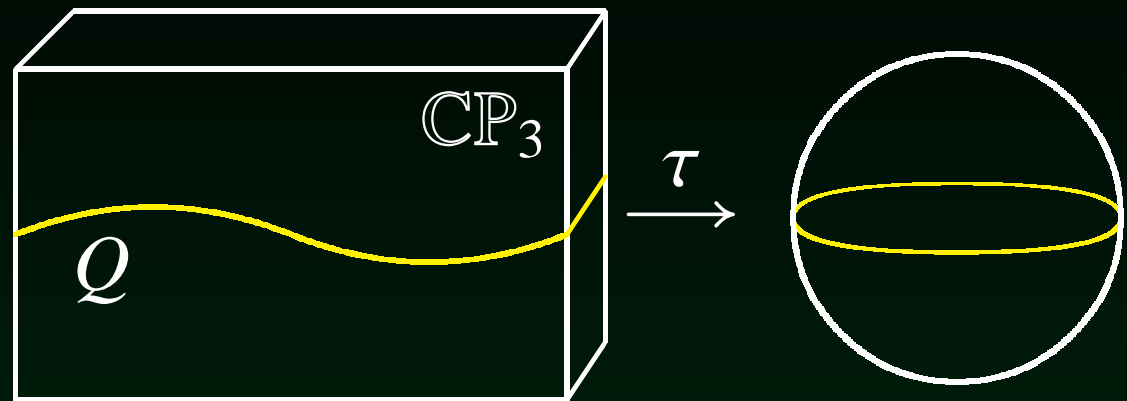
$$\begin{aligned}
 Q &= \{ [z] \in \mathbb{C}P_3 \mid \Re(z_2 \bar{z}_3 + z_4 \bar{z}_1) = 0 \} \\
 &\cong \{ [Z] \in \mathbb{C}P_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2 \} \\
 &\equiv \text{Levi-indefinite hyperquadric}
 \end{aligned}$$



(cf. saddle)

Twistor results

$$\begin{array}{ccc}
 Q & \subset & \mathbb{C}P_3 \\
 \downarrow & & \downarrow \tau \\
 S^3 & \subset & S^4
 \end{array}$$



Theorem A section $S^4 \supseteq \text{open } \Omega \xrightarrow{J} \mathbb{C}P_3$ of τ defines an integrable Hermitian structure if and only if $\tilde{M} \equiv J(\Omega)$ is a complex submanifold.

Theorem A section $S^3 \supseteq \text{open } \Omega \xrightarrow{U} Q$ of $\tau : Q \rightarrow S^3$ defines a conformal foliation if and only if $M \equiv U(\Omega)$ is a CR submanifold.

CR submanifolds and functions

$M \subset Q \subset \mathbb{C}\mathbb{P}_3$ is a ‘CR submanifold’?

It means: $TM \cap JTQ$ is preserved by J .

It does not mean: $M = \{f = 0\}$ where f is a

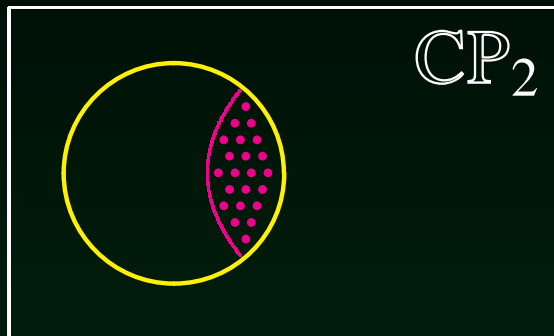
CR function: $(X + iJX)f = 0 \quad \forall X \in \Gamma(TQ \cap JTQ)$.

Implicit function theorem
is false in the CR category

- CR functions on Q are real-analytic.
- conformal foliations on S^3 need not be.

CR functions

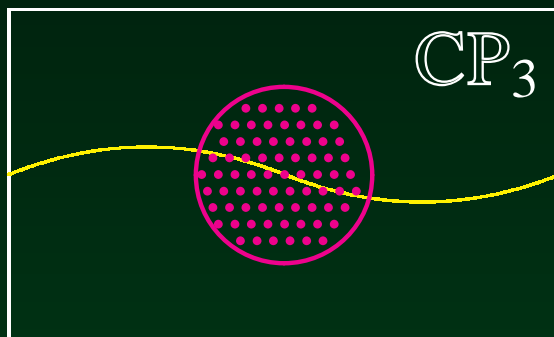
$$\{[Z] \in \mathbb{C}\mathbb{P}_2 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2\} = \text{three-sphere}$$



Theorem (H. Lewy 1956)

CR \Rightarrow holomorphic extension

$$\{[Z] \in \mathbb{C}\mathbb{P}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} = Q$$



Corollary

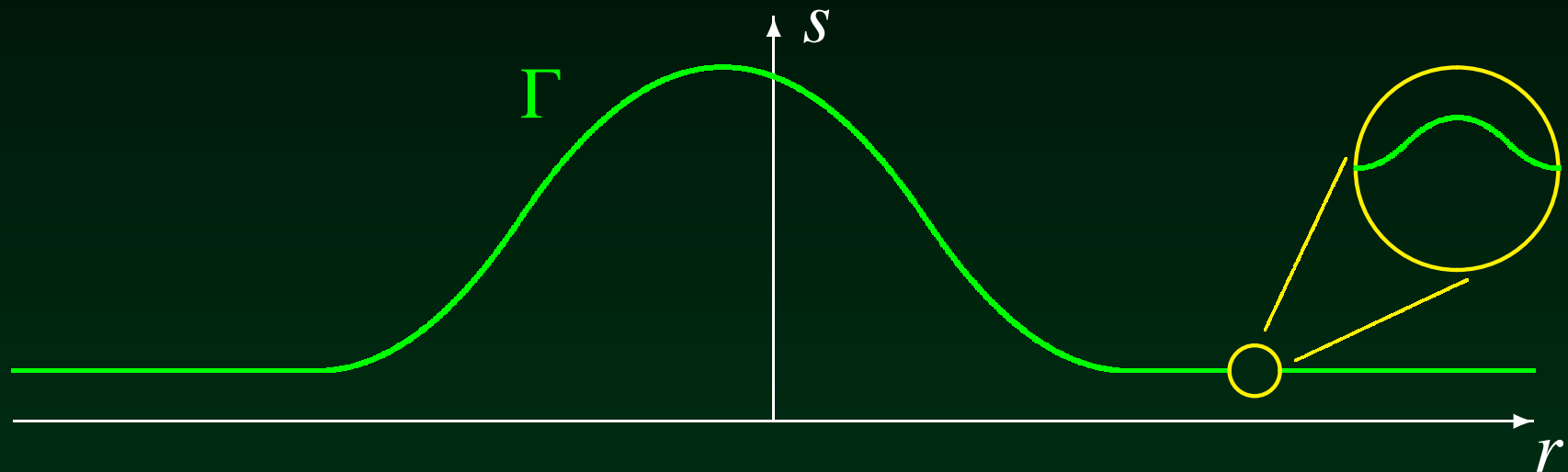
CR \Rightarrow holomorphic extension

Hence, a CR function on Q is real-analytic!

Smooth conjugate functions

Eikonal equation: $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial s}\right)^2 = 1$

Plenty of non-analytic solutions:



$f =$ signed distance to Γ

$$\left. \begin{array}{l} f(q, r, s) = f(r, s) \\ g(q, r, s) = q \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{array} \right.$$

QED

Real-analytic refinements

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.

Choose ω a \mathbb{C} -valued null 1-form on Ω s.t. $U \lrcorner \omega = 0$.

Lemma U conformal $\Leftrightarrow \omega \wedge d\omega = 0$.

Consider $h = f + ig : \Omega \rightarrow \mathbb{C}$ and let $\omega = dh$.

Remark • $d\omega = 0$ • f and g conjugate $\Leftrightarrow \omega$ is null.

Theorem Suppose $\omega : \Omega \rightarrow \mathbb{C}$ is real-analytic null.

• $\omega \wedge d\omega = 0 \Leftrightarrow \tilde{M} \hookrightarrow \mathbb{C}P_3$ (s.t. $M = \tilde{M} \cap Q$)

• $\omega^\perp \wedge d\omega = 0 \Leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ s.t. $\pi(\tilde{S}) = \tilde{M}$

• $d\omega = 0 \Leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ s.t. \tilde{S} is Lagrangian

Holomorphic function of two complex variables

THANK YOU