

Integral geometry on complex projective space

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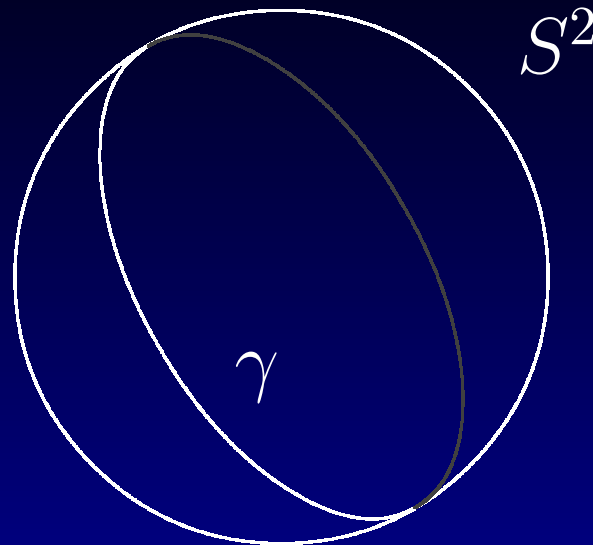
Topics

- Funk or Radon transforms on S^2 or \mathbb{R}^2
- X-ray transform on $\mathbb{R}P_3$
- Penrose transform on $\mathbb{C}P_3$
- Other Penrose transforms
- Complex v. Real (complex methods)
- X-ray transform on $\mathbb{C}P_3$
- Real v. Complex (real methods)
- Other double fibration transforms

Funk-Radon

- Funk (1913)

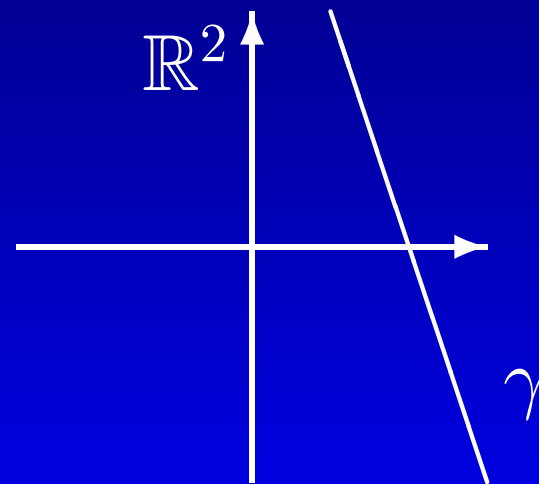
$$f \in \Gamma(S^2, \mathcal{E})$$



$$\phi(\gamma) = \int_{\gamma} f$$

- Radon (1917)

$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$



$$\phi(\gamma) = \int_{\gamma} f$$

Radon=Funk!

$$\mathcal{F} : \Gamma_{\text{even}}(S^2, \mathcal{E}) \xrightarrow{\cong} \Gamma_{\text{even}}(S^2, \mathcal{E})$$

$$\parallel \qquad \qquad \qquad \parallel$$

Better: $\Gamma(\mathbb{RP}_2, \mathcal{E}) \qquad \qquad \qquad \Gamma(\mathbb{RP}_2, \mathcal{E})$

Better still: $\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\cong} \Gamma(\mathbb{RP}_2^*, \tilde{\mathcal{E}}(-1))$

Usual affine coördinates $\mathbb{R}^2 \hookrightarrow \mathbb{RP}_2 \rightsquigarrow$

$$\begin{array}{ccc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} & \\ \downarrow & & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} & \end{array} \left. \vphantom{\begin{array}{ccc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} & \\ \downarrow & & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} & \end{array}} \right\} \text{agree!}$$

John (1938)

The **X-ray transform** according to John

$$\Gamma_*(\mathbb{R}^3, \mathcal{E}) \ni f \mapsto \phi(\gamma) = \int_{\gamma} f$$

Better: $\Gamma(\mathbb{R}\mathbb{P}_3, \mathcal{E}(-2)) \ni f \mapsto \phi(\gamma) = \oint_{\gamma} f$

NB invariance under $SL(4, \mathbb{R})$ because

$$\Gamma(\mathbb{R}\mathbb{P}_1, \mathcal{E}(-2)) \cong \Gamma(\mathbb{R}\mathbb{P}_1, \Lambda^1) \xrightarrow{\int} \mathbb{R}$$

is invariant under $SL(2, \mathbb{R})$.

X-ray transform on \mathbb{RP}_3

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \longrightarrow \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}[-1])$$

Range?

Theorem (John)

$$\phi = \mathcal{X}f \iff \square\phi = 0$$

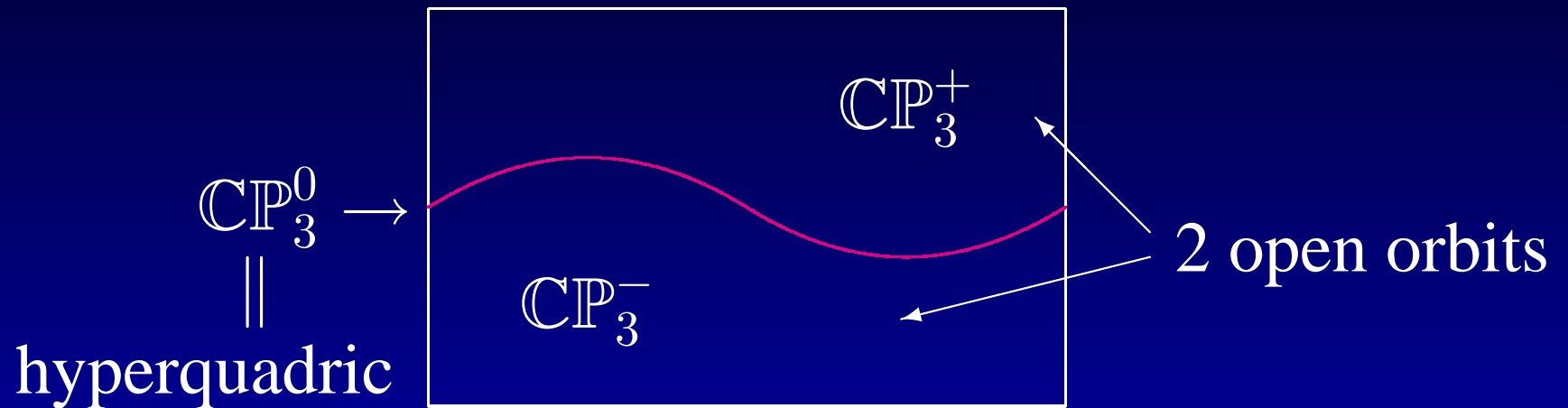
where

$$\square : \tilde{\mathcal{E}}[-1] \longrightarrow \tilde{\mathcal{E}}[-3]$$

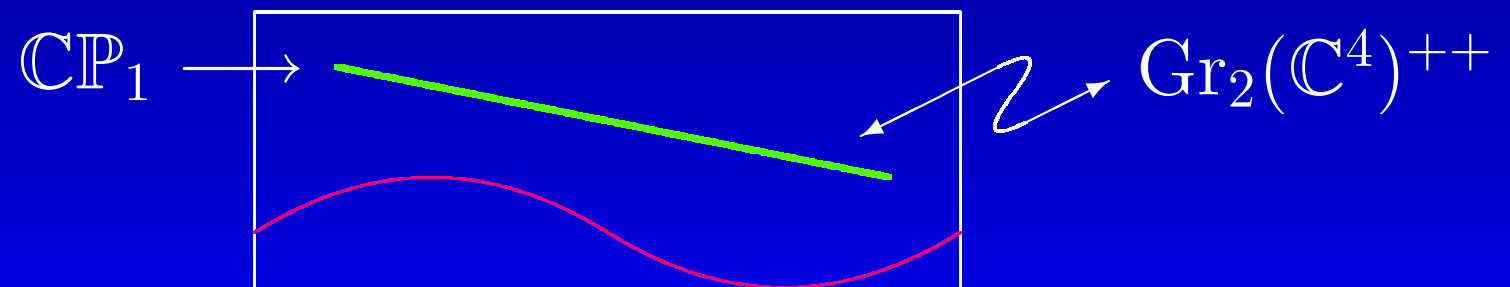
= ultrahyperbolic wave operator ($\text{SL}(4, \mathbb{R})$ -invariant).

Penrose (1978–81)

$SU(2, 2)$ acting on $\mathbb{C}P_3$ has 3 orbits:–



$SU(2, 2)$ acting on $\text{Gr}_2(\mathbb{C}^4)$ has 6 orbits (3 open):–



Penrose transform

$$\mathcal{P} : H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{C}^4)^{++}, \mathcal{O}[-1])$$

$$\text{NB: } H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \cong H^1(\mathbb{CP}_1, \Omega^1) \xrightarrow{\int} \mathbb{C}$$

Range?

Theorem (E-Penrose-Wells (1981))

$$\phi = \mathcal{P}\omega \iff \square\phi = 0$$

where

$$\square : \mathcal{O}[-1] \rightarrow \mathcal{O}[-3]$$

= holomorphic wave operator ($\mathrm{SL}(4, \mathbb{C})$ -invariant).

Twistor transform (cf. Fourier)

$$\mathcal{T} : H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \xrightarrow{\cong} H^1(\mathbb{CP}_3^{*-}, \mathcal{O}(-2))$$

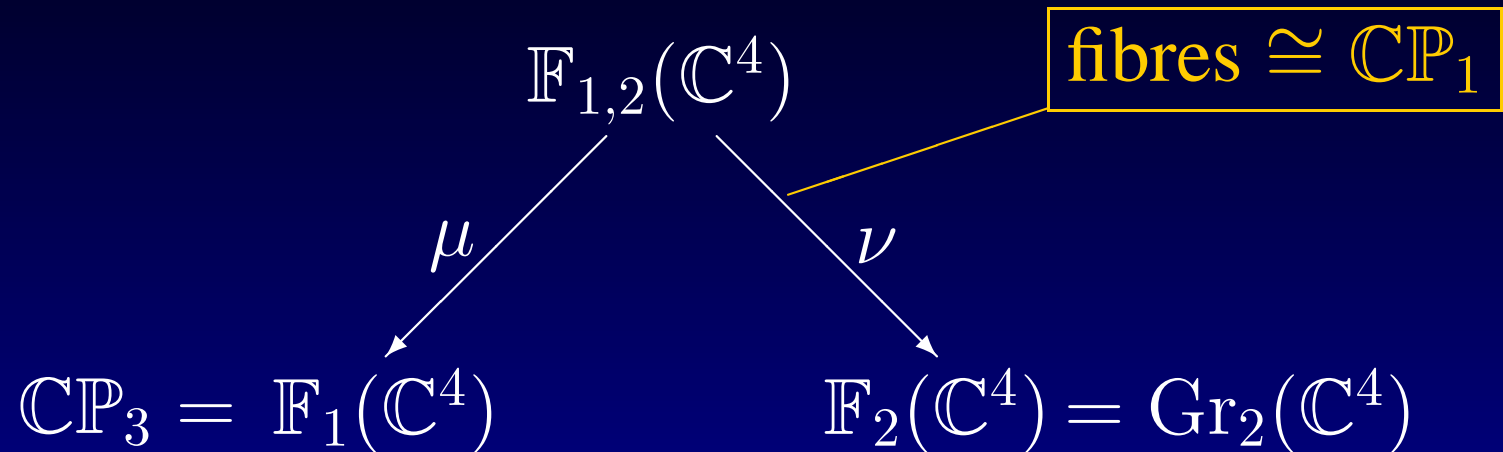
both \cong { +ve-frequency **hyperfunction** solutions of the wave-equation on compactified Minkowski space }.

$$\begin{array}{ccc} \mathcal{T} : H^1(|\mathbb{CP}_3^+|, \mathcal{O}(-2)) & \xrightarrow{\cong} & H^1(|\mathbb{CP}_3^{*-}|, \mathcal{O}(-2)) \\ & & \downarrow \cong \\ & & H_*^2(\mathbb{CP}_3^{*+}, \mathcal{O}(-2)) \end{array}$$

\cong **real-analytic** solutions. Defines **unitary structure**:–

$$\langle \alpha, \beta \rangle \equiv \int_{\mathbb{CP}_3^+} \alpha \wedge \overline{\mathcal{T}\beta}$$

Double fibration



$$\mathbb{C}\mathbb{P}_3^+ = \mu(\nu^{-1}(\text{Gr}_2(\mathbb{C}^4)^{++}))$$

Spectral sequence (cf. Leray)

$$\begin{aligned} E_1^{p,q} &= \Gamma(\text{Gr}_2(\mathbb{C}^4)^{++}, \nu_*^q(\Omega_\mu^p \otimes \mu^*V)) \\ &\implies H^{p+q}(\mathbb{C}\mathbb{P}_3^+, \mathcal{O}(V)) \end{aligned}$$

Twistor fibration (cf. Hopf)

$$\begin{array}{ccc} \mathbb{C}\mathbb{P}_3 & \mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) & \\ \tau \downarrow & & \rightarrow \Gamma(U, \mathcal{E}[-1]) \\ S^4 \supseteq \text{open } U & & \end{array}$$

Range?

$$\phi = \mathcal{P}\omega \iff \square\phi = 0$$

where

$$\square : \mathcal{E}[-1] \rightarrow \mathcal{E}[-3]$$

= conformal Laplacian (SO(5, 1)-invariant).

Other Penrose transforms

$$\begin{array}{ccc} Z & & \text{Atiyah-Hitchin-Singer (1978-81)} \\ \tau \downarrow & & \\ M & \leftarrow & \text{anti-self-dual 4-manifold} \end{array}$$

NB: a **conformally invariant** construction.

$$\mathcal{P} : H^1(Z, \kappa^{1/2}) \cong \{ \phi \in \Gamma(M, \mathcal{E}[-1]) \text{ s.t. } \square \phi = 0 \},$$

where $\square =$ **conformal Laplacian** $\Delta + R/6$.

$$\mathcal{P} : H^1(Z, \kappa) \cong \{ \phi \in \Gamma(M, \Lambda_-^2) \text{ s.t. } d\phi = 0 \}$$

= anti-self-dual Maxwell fields on M .

Example (Fubini-Study)

$\mathbb{C}\mathbb{P}_2$ (with opposite orientation) is anti-self-dual

$$\begin{array}{ccc} \mathbb{F}_{1,2}(\mathbb{C}^3) & \ni & L \subset P \\ \tau \downarrow & & \tau \downarrow \\ \mathbb{C}\mathbb{P}_2 & \ni & L^\perp \cap P \end{array}$$

Conclusions

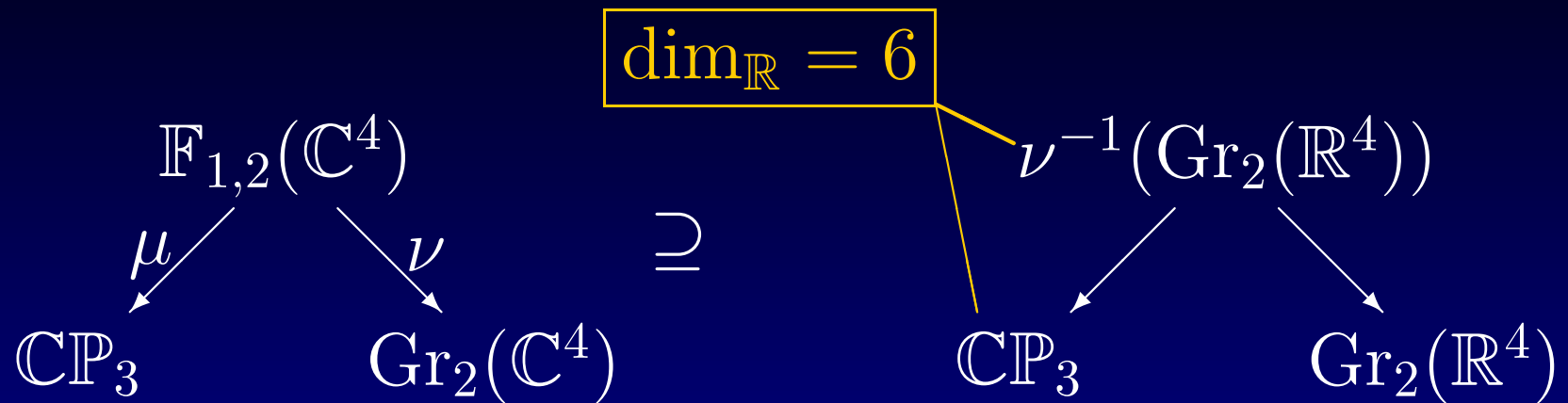
$H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}) = 0 \Rightarrow$ exactness of

$$\Gamma(\mathbb{C}\mathbb{P}_2, \Lambda^0) \rightarrow \Gamma(\mathbb{C}\mathbb{P}_2, \Lambda^1) \rightarrow \Gamma(\mathbb{C}\mathbb{P}_2, \Lambda_+^2).$$

$H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) = 0 \Rightarrow$ exactness of

$$\Gamma(\mathbb{C}\mathbb{P}_2, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{C}\mathbb{P}_2, \odot_o \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{C}\mathbb{P}_2, \boxplus_{o+}).$$

Complex v. Real (complex methods)



\rightsquigarrow fibration

$$\tau : \mathbb{C}\mathbb{P}_3 \setminus \mathbb{R}\mathbb{P}_3 \ni [Z] \longmapsto [iZ \wedge \bar{Z}] \in \text{Gr}_2(\mathbb{R}^4)$$

Calculation \implies

$$\Gamma(\text{Gr}_2, \tilde{\mathcal{E}}(-1)) \ni \phi \xrightarrow{\square} 0 \iff \underbrace{\bar{\partial} \partial \tau^* \phi}_{\Gamma(\mathbb{C}\mathbb{P}_3 \setminus \mathbb{R}\mathbb{P}_3, \Lambda^{0,1}(-2))} = 0 !!!$$

But (see later)

- $H^0(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0,$
- $H^1(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0.$

Therefore, if $\square\phi = 0$ then

$$\exists! F \in \Gamma(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{E}(-2)) \text{ s.t. } \bar{\partial}F = \partial\tau^*\phi.$$

Theorem (Bailey-E-Gover-Mason (1994–2003))

$$\phi = \mathcal{X}f \quad \Rightarrow \quad f = \text{‘lim’ } F.$$

$$\lim \Re F = f \quad \lim \Im F = \text{Hilbert transform of } f.$$

Real-analytic version (Aryapoor (2009)):-

$$\Gamma(\mathbb{RP}_3, \mathcal{O}(-2)) \xrightarrow{\cong} H_*^1(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) \xrightarrow{\cong} \{\phi \in \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{O}}(-1)) \text{ s.t. } \square\phi = 0\}.$$

X-ray transform on $\mathbb{C}\mathbb{P}_3$

$$\begin{array}{lll}
 0) & \mathcal{X} : \Gamma(\mathbb{C}\mathbb{P}_3, \Lambda^0) \ni f \mapsto & \gamma \mapsto \oint_{\gamma} f \\
 1) & \mathcal{X} : \Gamma(\mathbb{C}\mathbb{P}_3, \Lambda^1) \ni \omega \mapsto & \gamma \mapsto \oint_{\gamma} \omega \\
 2) & \mathcal{X} : \Gamma(\mathbb{C}\mathbb{P}_3, \odot^2 \Lambda^1) \ni h_{ab} \mapsto & \gamma \mapsto \oint_{\gamma} h_{ab} \\
 \vdots & & \vdots
 \end{array}$$

where $\gamma \in \{\text{Fubini-Study geodesics}\}$. **Kernel?**

- 0) \mathcal{X} is injective
- 1) $\mathcal{X}\omega = 0 \iff \omega = df$
- 2) $\mathcal{X}h_{ab} = 0 \iff h_{ab} = \nabla_{(a} X_{b)}$
- $\vdots \mathcal{X}h_{ab\dots d} = 0 \iff h_{ab\dots d} = \nabla_{(a} X_{b\dots d)}$

Tsukamoto (1981)

Real v. Complex (real methods)

- Bootstrap from $\mathbb{R}P_2$ to $\mathbb{C}P_2$ (E (1997))
- Bootstrap from $\mathbb{C}P_2$ to $\mathbb{C}P_3$ (Tsukamoto (1981))
- Bootstrap from $\mathbb{R}P_3$ to $\mathbb{C}P_3$ (E-Goldschmidt (...))

Geodesics? $\gamma \subset \mathbb{R}P_2 \subset \mathbb{C}P_2$ and act by $SU(3)$.

Example

$$\begin{aligned} \oint_{\gamma} \omega = 0, \quad \forall \gamma &\Rightarrow d\omega|_{\mathbb{R}P_2} = 0, \quad \forall \mathbb{R}P_2 \\ &\Leftrightarrow d\omega = fJ \quad (\text{J=Kähler form}) \\ &\Rightarrow d_+\omega = 0. \quad \text{However,} \end{aligned}$$

$\Gamma(\mathbb{C}P_2, \Lambda^0) \rightarrow \Gamma(\mathbb{C}P_2, \Lambda^1) \rightarrow \Gamma(\mathbb{C}P_2, \Lambda^2_+) \text{ is exact!}$

Proof of Tsukamoto

Similarly, to prove

$$\oint_{\gamma} h_{ab} = 0, \quad \forall \gamma \Rightarrow h_{ab} = \nabla_{(a} X_{b)}$$

on $\mathbb{C}\mathbb{P}_2$, we may use the **real** ingredients:–

- it's true on $\mathbb{R}\mathbb{P}_2$,
- $\Gamma(\mathbb{C}\mathbb{P}_2, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{C}\mathbb{P}_2, \odot_{\circ}^2 \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{C}\mathbb{P}_2, \boxplus_{\circ+})$
is exact.

These, in turn, may be proved by **complex** methods!

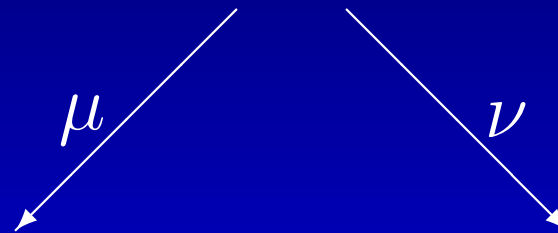
More double fibrations

Missing link: $H^r(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = ?$

Use SCV (Laufer's Serre Duality ...) or

$$\begin{aligned} \text{+ve-definite real quadrics} &\cong \mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4) \\ &\subset \mathrm{SL}(4, \mathbb{C})/\mathrm{SO}(4, \mathbb{C}) \end{aligned}$$

$$\mathrm{SL}(4, \mathbb{C}) / (P \cap \mathrm{SO}(4, \mathbb{C}))$$



$$\mathbb{CP}_3 = \mathrm{SL}(4, \mathbb{C}) / P$$

$$\mathrm{SL}(4, \mathbb{C}) / \mathrm{SO}(4, \mathbb{C})$$

$$\cup$$

$$\mathbb{CP}_3 \setminus \mathbb{RP}_3$$

Now transform (E-Wolf (...))...

$H^r(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0$ for $r = 0, 1$, **as required.**

THANK YOU

HAPPY BIRTHDAY BENT!