Some remarks on linear elasticity

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Stress and strain

in $\mathbb{R}^3$

Saint-Venant operator

displacement X_i

$\downarrow$

strain $\nabla_i X_j + \nabla_j X_i$ $\qquad \Sigma_{ij}$

$\downarrow$

stress $\epsilon_i{}^{km}\epsilon_j{}^{\ell n}\nabla_k\nabla_\ell\Sigma_{mn}$ $\qquad G_{ij}$

$\downarrow$

load $\nabla^i G_{ij}$

cf. de Rham

$$f \xrightarrow{\text{grad}} \nabla_i f \qquad \omega_i \xrightarrow{\text{curl}} \epsilon_i{}^{jk}\nabla_j\omega_k \qquad \phi_i \xrightarrow{\text{div}} \nabla^i\phi_i$$

Linear elasticity complex

Elasticity $\Lambda^1 \xrightarrow{\text{Killing}} \bigodot^2\Lambda^1 \xrightarrow{\text{St-Venant}} \bigodot^2\Lambda^1 \xrightarrow{\text{Bianchi}} \Lambda^1$

de Rham $\Lambda^0 \xrightarrow{\text{grad}} \Lambda^1 \xrightarrow{\text{curl}} \Lambda^1 \xrightarrow{\text{div}} \Lambda^0$

Hesse $\Lambda^0 \xrightarrow{\text{Hessian}} \bigodot^2\Lambda^1 \longrightarrow \bigotimes^2_{\text{tf}}\Lambda^1 \longrightarrow \Lambda^1$

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$$\sigma \longmapsto \nabla_i\nabla_j\sigma \qquad \psi_{ij} \longmapsto \epsilon_i{}^{k\ell}\nabla_k\psi_{j\ell} \qquad \theta_{ij} \longmapsto \nabla^i\theta_{ij}$$

Riemannian geometry

In three-dimensions

metric $\qquad g_{ij} + t\,\Sigma_{ij}$

$$\updownarrow$$

Ricci curvature $\qquad R_{ij} + t\ \ldots$

$$\Updownarrow$$

Einstein tensor $\qquad G_{ij} + t\,\epsilon_i{}^{km}\epsilon_j{}^{\ell n}\nabla_k\nabla_\ell\Sigma_{mn} + \cdots$

$$G_{ij} = Rg_{ij} - 2R_{ij}$$

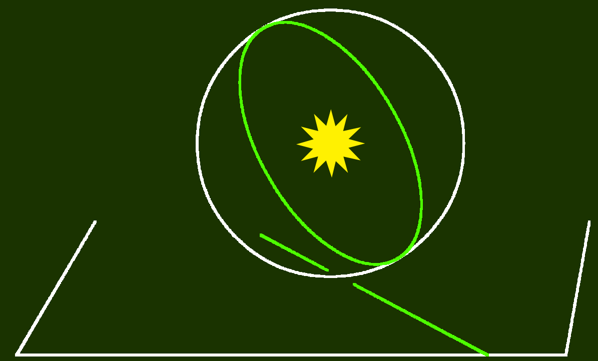
$$\nabla^i G_{ij} = 0$$

nonlinear 'complex'

$$\left\{\begin{array}{l}\text{coördinate}\\ \text{freedom}\end{array}\right\} \to \{\text{metrics}\} \to \{\text{curvature}\} \xrightarrow{\text{Bianchi}} \Lambda^1$$

Projective geometry

$\mathbb{R}^3 \hookrightarrow S^3 \subset \mathbb{R}^4$ by central projection



On $S^3 \subset \mathbb{R}^4$

$$
\begin{array}{ccccccc}
\Lambda^0 \otimes \mathbb{R}^4 & \xrightarrow{\text{grad}} & \Lambda^1 \otimes \mathbb{R}^4 & \xrightarrow{\text{curl}} & \Lambda^1 \otimes \mathbb{R}^4 & \xrightarrow{\text{div}} & \Lambda^0 \otimes \mathbb{R}^4 \\
\| & & \| & & \| & & \| \\
\Lambda^0 & & \Lambda^1 & & \Lambda^1 & & \Lambda^0 \\
\oplus & \stackrel{\cong}{\nearrow} & \oplus & \stackrel{\mu_{ij} \mapsto \epsilon_i{}^{jk} \mu_{jk}}{\nearrow} & \oplus & \stackrel{\nu_{ij} \mapsto \nu^i{}_i}{\nearrow} & \oplus \\
\Lambda^1 & & \bigotimes^2 \Lambda^1 & & \bigotimes^2 \Lambda^1 & & \Lambda^1
\end{array}
$$

$\rightsquigarrow$ Hesse complex $\Lambda^0 \xrightarrow{\text{Hessian}} \bigodot^2 \Lambda^1 \longrightarrow \bigotimes^2_{\text{tf}} \Lambda^1 \longrightarrow \Lambda^1$

$\rightsquigarrow$ new stable finite element schemes (Arnold et alia)

Further reading

- D.N. Arnold, R.S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica 15 (2006) 1–155.
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- P.G. Ciarlet, *Mathematical Elasticity vol 1: Three-dimensional Elasticity*, North-Holland 1988.
- M.G. Eastwood, *Ricci curvature and the mechanics of solids*, Austral. Math. Soc. Gaz. 37 (2010) 238–241.

THANK YOU