# Projective space and twistor theory


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# Topics

- $\mathbb{CP}_3$ is the twistor space of $S^4$,
- Penrose transform on $\mathbb{CP}_3$,

classical twistor theory

& CR geometry

- Funk-Radon transform on $\mathbb{RP}_2$,
- X-ray transform on $\mathbb{RP}_3$,

with round metric

- X-ray transform on $\mathbb{CP}_2$,
- Penrose transform on $\mathbb{CP}_2$,
- X-ray transform on $\mathbb{CP}_3$,

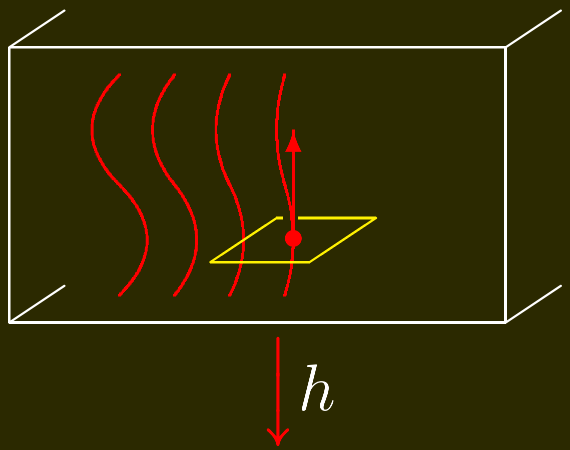
with Fubini-Study metric

- BGG-like complexes on $\mathbb{CP}_3$.

Bernstein-Gelfand-Gelfand

# Conformal foliations

$U$ = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.




$\mathbb{C}$

$U$ is (transversally) conformal $\Leftrightarrow \mathcal{L}_U$ preserves the conformal metric orthogonal to its leaves

isothermal coördinates

$$h = f + ig \qquad \langle \nabla f, \nabla g \rangle = 0, \quad \|\nabla f\| = \|\nabla g\|$$

conjugate functions

# Conjugate functions on $\mathbf{R}^3$

$$f = f(q,r,s) \quad g = g(q,r,s) \quad \text{s.t.} \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $\boxed{f = r \quad g = s}$
- $\boxed{f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2+s^2}}$
- $\boxed{f = r\dfrac{q^2+r^2+s^2}{r^2+s^2} \quad g = s\dfrac{q^2+r^2+s^2}{r^2+s^2}}$
- $$\boxed{\begin{aligned} f &= \frac{(1-q^2-r^2-s^2)r + 2qs}{r^2+s^2} \\ g &= \frac{(1-q^2-r^2-s^2)s - 2qr}{r^2+s^2} \end{aligned}}$$

$$\begin{array}{ccc} \mathbb{R}^3 \hookrightarrow & S^3 & \\ & \downarrow & \text{Hopf} \\ \mathbb{R}^2 \leftarrow & S^2 \setminus \{*\} & \end{array}$$

# Almost Hermitian structures

NB: $J(p,q,r,s) : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies

- $J^2 = -\mathrm{Id}$
- $J \in \mathrm{SO}(4)$

$$\iff J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & o & -u \\ w & -v & u & 0 \end{bmatrix}$$

$u^2 + v^2 + w^2 = 1,$ two-sphere

Consider $\mathbb{R}^3 = \{(p,q,r,s) \in \mathbb{R}^4 \mid p = 0\} \subset \mathbb{R}^4$

NB: $U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3} = \left(u\frac{\partial}{\partial q} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial s}\right)\Big|_{\mathbb{R}^3}$

unit vector field

also $\rightsquigarrow$ two-sphere

# Sphere bundles

bundle of unit vectors

bundle of almost Hermitian structues

$$
\begin{array}{ccc}
Q_\circ & \subset & Z_\circ \\
\big\downarrow & & \tau\big\downarrow \\
\mathbb{R}^3 & \subset & \mathbb{R}^4
\end{array}
$$

section

$\updownarrow$

unit vector field

section

$\updownarrow$

almost Hermitian structure

# Hermitian structures

Lemma

$J$ is integrable $\Longrightarrow U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3}$ is conformal

Conversely??

NB: $J$ integrable $\Longrightarrow J$ real-analytic

Question: $U$ conformal $\Longrightarrow U$ real-analytic??

Answer: $\boxed{\boxed{\text{NO!}}}$

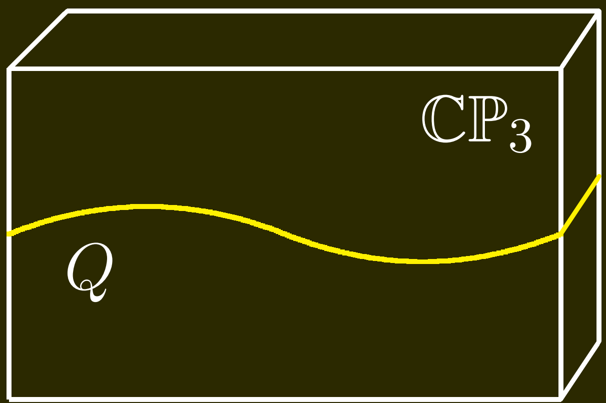
However: $U$ real-analytic and conformal
$\Longrightarrow U$ extends uniquely to an integrable $J$.

WHY?

# Twistor geometry

$$\begin{array}{ccc} Q_\circ & \subset & Z_\circ \\ \downarrow & & \downarrow \\ \mathbb{R}^3 & \subset & \mathbb{R}^4 \end{array} \quad \overset{\text{compactify}}{\hookrightarrow} \quad \begin{array}{ccc} Q & \subset & \mathbb{CP}_3 \\ \downarrow & & \tau\downarrow \\ S^3 & \subset & S^4 \end{array}$$

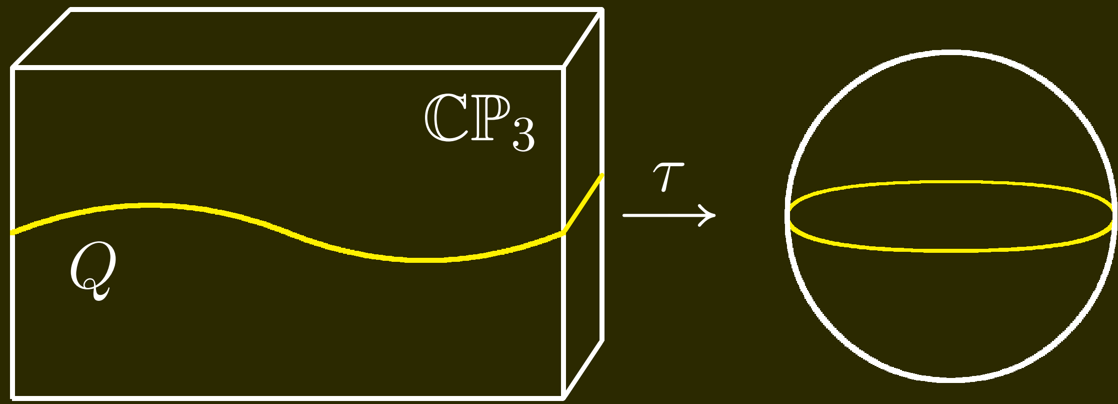
$$\begin{aligned} Q &= \{[Z] \in \mathbb{CP}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} \\ &\equiv \text{Levi-indefinite hyperquadric} \end{aligned}$$




(cf. saddle)

# Twistor results

$$\begin{tikzcd} Q \arrow[d] & \subset & \mathbb{CP}_3 \arrow[d, "\tau"'] \\ S^3 & \subset & S^4 \end{tikzcd}$$




Theorem A section $S^4 \supseteq^{\text{open}} \Omega \xrightarrow{J} \mathbb{CP}_3$ of $\tau$ defines an integrable Hermitian structure if and only if $\widetilde{M} \equiv J(\Omega)$ is a complex submanifold.

Theorem A section $S^3 \supseteq^{\text{open}} \Omega \xrightarrow{U} Q$ of $\tau : Q \to S^3$ defines a conformal foliation if and only if $M \equiv U(\Omega)$ is a CR submanifold.

# CR submanifolds and functions

$M \subset Q \subset \mathbb{CP}_3$ is a 'CR submanifold'?

It means: $TM \cap JTQ$ is preserved by $J$.

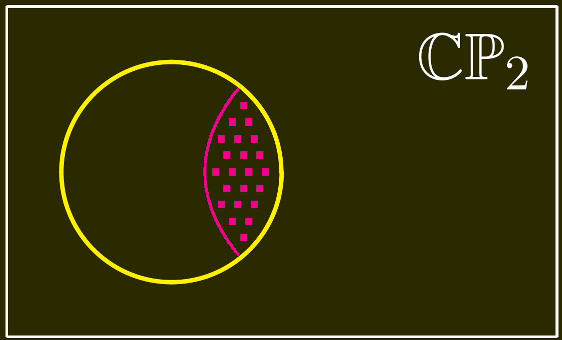
It does not mean: $M = \{f = 0\}$ where $f$ is a CR function: $(X + iJX)f = 0 \ \ \forall X \in \Gamma(TQ \cap JTQ)$.

Implicit function theorem
is false in the CR category

- CR functions on $Q$ are real-analytic.
- conformal foliations on $S^3$ need not be.

# CR functions

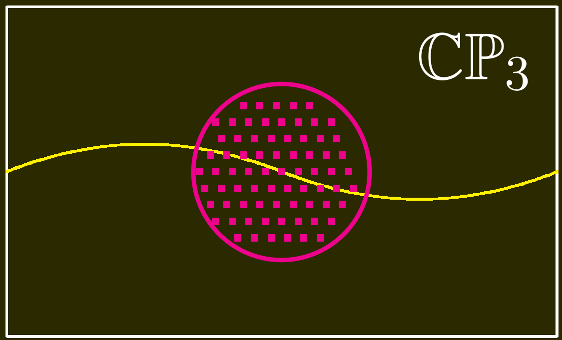
$$\{[Z] \in \mathbb{CP}_2 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2\} = \text{ three-sphere}$$




Theorem (H. Lewy 1956)

CR $\Rightarrow$ holomorphic extension

$$\{[Z] \in \mathbb{CP}_3 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} = Q$$




Corollary

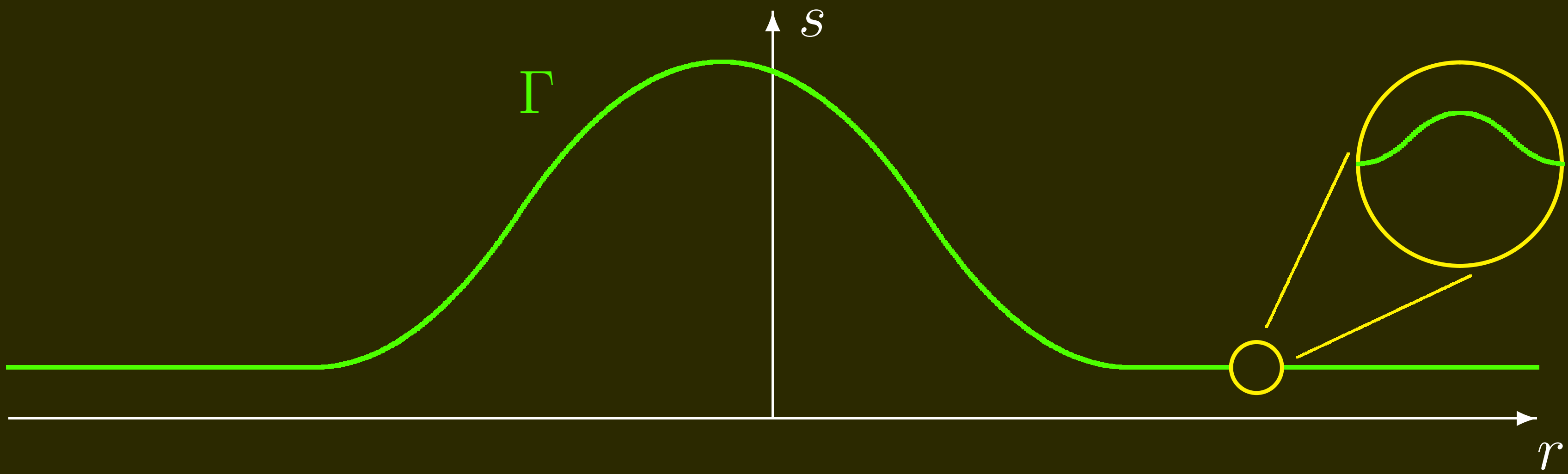
CR $\Rightarrow$ holomorphic extension

Hence, $\boxed{\text{a CR function on } Q \text{ is real-analytic!}}$

# Smooth conjugate functions

Eikonal equation: $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial s}\right)^2 = 1$

Plenty of non-analytic solutions:




$f$ = signed distance to $\Gamma$

$$\left.\begin{array}{rcl} f(q,r,s) & = & f(r,s) \\ g(q,r,s) & = & q \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{array}}$$

QED

# Penrose transform

$$\begin{array}{ccccc}
\tau^{-1}(U) & \subseteq & \mathbb{CP}_3 & & H^1(\tau^{-1}(U), \mathcal{O}(-2)) \\
\downarrow & & \tau\downarrow & \rightsquigarrow & \downarrow\wr \\
U^{\text{open}} & \subseteq & S^4 & & \{\phi : U \to \mathbb{C} \mid \underbrace{(\Delta - R/6)}_{\text{conformal Laplacian}}\phi = 0\}
\end{array}$$

homogeneous vector bundle

conformal Laplacian

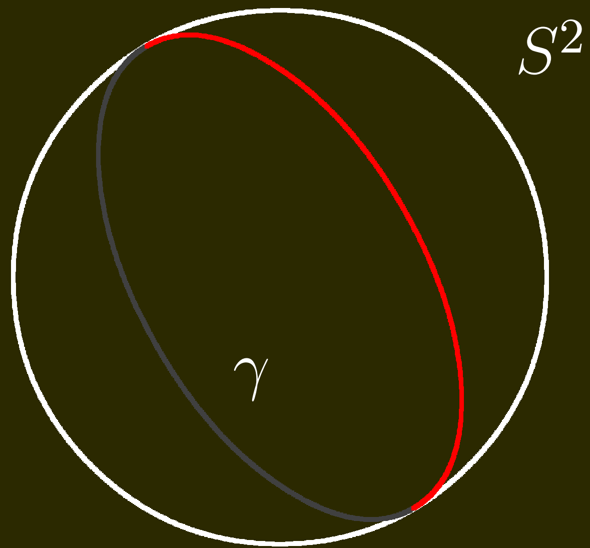
$$\begin{array}{ccccc}
\mathbb{F}_{1,2}(\mathbb{C}^3) & \ni & L \subset P & & H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) \\
\tau\downarrow & & \downarrow & \rightsquigarrow & \downarrow\wr \\
\mathbb{CP}_2 & \ni & L^\perp \cap P & & \dfrac{\Gamma(\mathbb{CP}_2, \odot^2_\circ \Lambda^1) \to \Gamma(\mathbb{CP}_2, \boxplus^{2,2}_{\circ\perp} \Lambda^1)}{\Gamma(\mathbb{CP}_2, \Lambda^1) \to \Gamma(\mathbb{CP}_2, \odot^2_\circ \Lambda^1)}
\end{array}$$

Fubini-Study metric

# Funk-Radon transform on $\mathbb{RP}_2$

$$\begin{tikzcd}
 & \mathbb{F}_{1,2}(\mathbb{R}^3) \arrow[dl] \arrow[dr] & \\
\mathbb{R}^2 \hookrightarrow \mathbb{RP}_2 & & \mathrm{Gr}_2(\mathbb{R}^3) = \mathbb{RP}_2^*
\end{tikzcd}$$

projective duality




$$\phi(\gamma) = \oint_\gamma f$$

$\mathrm{GL}(3,\mathbb{R})$-invariant

Theorem (Funk 1913) $C^\infty(\mathbb{RP}_2) \xrightarrow{\simeq} C^\infty(\mathbb{RP}_2^*)$

Better Theorem $\Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\simeq} \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1))$

# X-ray transform on $\mathbb{RP}_3$

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{F}_{1,2}(\mathbb{R}^4) \arrow[dl] \arrow[dr] & & \hookrightarrow & & \mathbb{F}_{1,2}(\mathbb{C}^4) \arrow[dl] \arrow[dr] & \\
\mathbb{RP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) & & \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{C}^4) \supset S^4
\end{tikzcd}
$$

$\boxed{\mathrm{GL}(4,\mathbb{R}) \hookrightarrow \mathrm{GL}(4,\mathbb{C})}$

$\boxed{\mathrm{Spin}(5,1)\text{-orbit}}$

Theorem (cf. John 1938)

$$
\Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\simeq} \ker : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1]) \xrightarrow{\square} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3])
$$

X-ray transform

ultra-hyperbolic wave operator

$$
\boxed{\widetilde{\square}|_{S^4} = \Delta - R/6}
$$

# Machinery for the X-ray transform

Complex analysis comes into play in two ways:

- constructing a spectral sequence,
- computing with the spectral sequence.

More generally:

$\boxed{\dim_{\mathbb{R}} = 2n}$

$$\begin{tikzcd} & \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & & \supset & & \nu^{-1}(\mathrm{Gr}_2(\mathbb{R}^{n+1})) \equiv F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{CP}_n & & \mathrm{Gr}_2(\mathbb{C}^{n+1}) & & \mathbb{CP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \end{tikzcd}$$

$$\cup$$

$$\begin{tikzcd} & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \arrow[dl] \arrow[dr] & \\ \mathbb{RP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \end{tikzcd}$$

a correspondence

☠ but not a double fibration ☠

# Real blow up

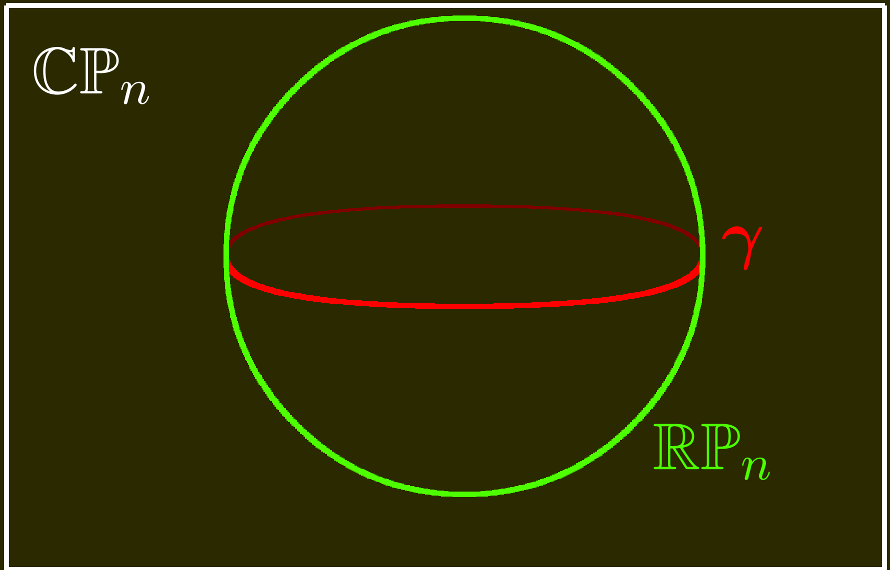
$$
\begin{array}{ccc}
F & = & \left\{ (L,P) \text{ s.t. } \begin{array}{l} L \subset \mathbb{C}^{n+1} \text{ is a complex line} \\ P \subset \mathbb{R}^{n+1} \text{ is a real plane} \\ \Re(L) \subseteq P \text{ (generic equality)} \end{array} \right\} \\
\downarrow \eta & & \downarrow \\
\mathbb{CP}_n & = & \{ L \text{ s.t. } L \subset \mathbb{C}^{n+1} \text{ is a complex line} \}
\end{array}
$$

$$
\boxed{\boxed{\begin{array}{ccc}
F & \supset & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \\
\downarrow \eta & & \downarrow \\
\mathbb{CP}_n & \supset & \mathbb{RP}_n
\end{array}}}
$$

Real blow up of $\mathbb{CP}_n$ along $\mathbb{RP}_n$ !

$F$ inherits an <u>involutive structure</u>

# X-ray transform on $\mathbb{CP}_2$ and $\mathbb{CP}_3$




$\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ induced by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is totally geodesic.

Translates by $\mathrm{SU}(n+1)$ too!

$\uparrow$ ‘Model Embeddings’ $\mu$

- The X-ray transform on $\mathbb{RP}_n$ is well-understood.
- Pullback of tensors under $\mu$ is well-understood.
- Suitable global techniques on $\mathbb{CP}_n$ are available:-

$n = 2$ ☹ Penrose transform of $H^1(\mathbb{F}_{1,2}(\mathbb{C}^2), \Theta) = 0$ etc.

$n \geq 3$ ☺ BGG-like: $0 \to \Lambda^1 \to \odot^2\Lambda^1 \to \boxplus_\perp \Lambda^1$ etc.

THE END

THANK YOU