Projective space and twistor theory

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• $\mathbb{CP}_3$ is the twistor space of $S^4$,

• Penrose transform on $\mathbb{CP}_3$,

• Funk-Radon transform on $\mathbb{RP}_2$,

• X-ray transform on $\mathbb{RP}_3$,

• X-ray transform on $\mathbb{CP}_2$,

• Penrose transform on $\mathbb{CP}_2$,

• X-ray transform on $\mathbb{CP}_3$,

• BGG-like complexes on $\mathbb{CP}_3$.

\{ classical twistor theory & CR geometry \}

\{ with round metric \}

\{ with Fubini-Study metric \}

Bernstein-Gelfand-Gelfand
Conformal foliations

\( U = \) unit vector field on \( \Omega^{\text{open}} \subseteq \mathbb{R}^3 \).

\( U \) is (transversally) conformal \iff \( \mathcal{L}_U \) preserves the conformal metric orthogonal to its leaves

Isothermal coördinates

\[ h = f + ig \quad \langle \nabla f, \nabla g \rangle = 0 \]
\[ \| \nabla f \| = \| \nabla g \| \]

Conjugate functions
Conjugate functions on $\mathbb{R}^3$

$$f = f(q, r, s) \quad g = g(q, r, s) \quad \text{s.t.} \quad \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \| \nabla f \| = \| \nabla g \| \end{cases}$$

- $f = r \quad g = s$
- $f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2 + s^2}$
- $f = r \frac{q^2 + r^2 + s^2}{r^2 + s^2} \quad g = s \frac{q^2 + r^2 + s^2}{r^2 + s^2}$
- $f = \frac{(1 - q^2 - r^2 - s^2)r + 2qs}{r^2 + s^2} \quad g = \frac{(1 - q^2 - r^2 - s^2)s - 2qr}{r^2 + s^2}$

$\mathbb{R}^3 \rightarrow S^3 \quad \downarrow \quad \text{Hopf}$

$\mathbb{R}^2 \leftarrow S^2 \setminus \{ * \}$
Almost Hermitian structures

NB: \( J(p, q, r, s) : \mathbb{R}^4 \to \mathbb{R}^4 \) satisfies

\[ J^2 = -\text{Id} \]

\[ J \in \text{SO}(4) \]

\[
J = \begin{bmatrix}
0 & -u & -v & -w \\
 u & 0 & -w & v \\
v & w & o & -u \\
w & -v & u & 0 \\
\end{bmatrix}
\]

\[ u^2 + v^2 + w^2 = 1, \text{ two-sphere} \]

Consider \( \mathbb{R}^3 = \{ (p, q, r, s) \in \mathbb{R}^4 \mid p = 0 \} \subset \mathbb{R}^4 \)

NB: \( U \equiv \left( J \frac{\partial}{\partial p} \right) \bigg|_{\mathbb{R}^3} = \left( u \frac{\partial}{\partial q} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial s} \right) \bigg|_{\mathbb{R}^3} \)

unit vector field

also \sim \text{ two-sphere}
Sphere bundles

bundle of unit vectors

$Q_0 \subset \mathbb{Z}_0 \subset \mathbb{R}^4$

bundle of almost Hermitian structures

section

unit vector field

section

almost Hermitian structure
Hermitian structures

Lemma

$J$ is integrable $\implies U \equiv \left( J \frac{\partial}{\partial p} \right) |_{\mathbb{R}^3}$ is conformal

Conversely??

NB: $J$ integrable $\implies J$ real-analytic

Question: $U$ conformal $\implies U$ real-analytic??

Answer: [NO!]

However: $U$ real-analytic and conformal $\implies U$ extends uniquely to an integrable $J$. WHY?
Twistor geometry

\[ Q \subset Z \subset \mathbb{R}^4 \]

compactify

\[ Q \subset \mathbb{C}P_3 \]

\[ S^3 \subset S^4 \]

\[ Q = \{ [Z] \in \mathbb{C}P_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2 \} \]

\[ \equiv \text{Levi-indefinite hyperquadric} \]

(cf. saddle)
Twistor results

\[ Q \subset \mathbb{CP}^3 \]
\[ S^3 \subset S^4 \]

Theorem A section \( S^4 \supset \text{open } \Omega \xrightarrow{J} \mathbb{CP}^3 \) of \( \tau \) defines an integrable Hermitian structure if and only if \( \tilde{M} \equiv J(\Omega) \) is a complex submanifold.

Theorem A section \( S^3 \supset \text{open } \Omega \xrightarrow{U} Q \) of \( \tau : Q \to S^3 \) defines a conformal foliation if and only if \( M \equiv U(\Omega) \) is a CR submanifold.
CR submanifolds and functions

$M \subset Q \subset \mathbb{CP}_3$ is a ‘CR submanifold’?

It means: $TM \cap JTQ$ is preserved by $J$.

It does not mean: $M = \{f = 0\}$ where $f$ is a CR function: $(X + iJX)f = 0 \ \forall X \in \Gamma(TQ \cap JTQ)$.

Implicit function theorem is false in the CR category

- CR functions on $Q$ are real-analytic.
- Conformal foliations on $S^3$ need not be.
CR functions

\[ \{ [Z] \in \mathbb{CP}_2 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 \} = \text{three-sphere} \]

**Theorem** (H. Lewy 1956)
CR \(\Rightarrow\) holomorphic extension

\[ \{ [Z] \in \mathbb{CP}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2 \} = Q \]

**Corollary**
CR \(\Rightarrow\) holomorphic extension

Hence, a CR function on \(Q\) is real-analytic!
Smooth conjugate functions

Eikonal equation: \[ \left( \frac{\partial f}{\partial r} \right)^2 + \left( \frac{\partial f}{\partial s} \right)^2 = 1 \]

Plenty of non-analytic solutions:

\[ f = \text{signed distance to } \Gamma \]

\[ f(q,r,s) = f(r,s) \]
\[ g(q,r,s) = q \]

\[ \langle \nabla f, \nabla g \rangle = 0 \]
\[ \| \nabla f \| = \| \nabla g \| \]

QED
Penrose transform

\[ \tau^{-1}(U) \subseteq \mathbb{CP}^3 \]

\[ \downarrow \]

\[ U_{\text{open}} \subseteq S^4 \]

\[ \sim \]

\[ H^1(\tau^{-1}(U), O(-2)) \]

\[ \sim \]

\[ \{ \phi: U \to \mathbb{C} | (\Delta - R/6)\phi = 0 \} \]

homogeneous vector bundle

conformal Laplacian

Fubini-Study metric

\[ F_{1,2}(\mathbb{C}^3) \ni L \subseteq P \]

\[ \tau \downarrow \]

\[ \downarrow \]

\[ \sim \]

\[ H^1(F_{1,2}(\mathbb{C}^3), \Theta) \]

\[ \sim \]

\[ \frac{\Gamma(\mathbb{CP}^2, \mathcal{O}_\mathbb{P}_2^2 \Lambda^1)}{\Gamma(\mathbb{CP}^2, \Lambda^1)} \to \frac{\Gamma(\mathbb{CP}^2, \oplus_{\mathbb{P}_2}^2 \Lambda^1)}{\Gamma(\mathbb{CP}^2, \mathcal{O}_\mathbb{P}_2^2 \Lambda^1)} \]
Funk-Radon transform on $\mathbb{RP}_2$

$\mathbf{F}_{1,2}(\mathbb{R}^3)$

$\mathbb{R}^2 \leftrightarrow \mathbb{RP}_2$

$\text{Gr}_2(\mathbb{R}^3) = \mathbb{RP}_2^*$

$\phi(\gamma) = \int_{\gamma} f$

$\Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\sim} \Gamma(\mathbb{RP}_2^*, \tilde{\mathcal{E}}(-1))$

$\text{GL}(3, \mathbb{R})$-invariant

Theorem (Funk 1913) $C^\infty(\mathbb{RP}_2) \xrightarrow{\sim} C^\infty(\mathbb{RP}_2^*)$
X-ray transform on $\mathbb{RP}_3$

Theorem (cf. John 1938)

$\Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\sim} \ker : \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}[-1]) \xrightarrow{} \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}[-3])$

X-ray transform

ultra-hyperbolic wave operator

$\tilde{\Box}|_{S^4} = \Delta - R/6$
Machinery for the X-ray transform

Complex analysis comes into play in two ways:

- constructing a spectral sequence,
- computing with the spectral sequence.

More generally:

\[
\begin{align*}
\dim_{\mathbb{R}} &= 2n \\
\mathbb{F}_{1,2}(\mathbb{C}^{n+1}) &\xrightarrow{\mu} \mathbb{C}P_n \\
\mathbb{G}r_2(\mathbb{C}^{n+1}) &\xrightarrow{\nu} \\
\mathbb{C}P_n &\supset \\
\mathbb{G}r_2(\mathbb{C}^{n+1}) &\xrightarrow{\nu^{-1}(\mathbb{G}r_2(\mathbb{R}^{n+1}))} F \\
\eta &\xrightarrow{\mathbb{F}_{1,2}(\mathbb{R}^{n+1})} \mathbb{C}P_n \\
\tau &\xrightarrow{\mathbb{G}r_2(\mathbb{R}^{n+1})} \\
\mathbb{R}P_n &\cup \\
\mathbb{G}r_2(\mathbb{R}^{n+1}) \\
\end{align*}
\]

\[\equiv\]

a correspondence

but not a double fibration

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Real blow up

\[ F = \begin{cases} (L, P) \quad \text{s.t.} \quad P \subset \mathbb{R}^{n+1} \text{ is a real plane} \\ L \subset \mathbb{C}^{n+1} \text{ is a complex line} \\ \mathcal{R}(L) \subseteq P \text{ (generic equality)} \end{cases} \]

\[ \eta \]

\[ \mathbb{CP}_n = \{ L \text{ s.t. } L \subset \mathbb{C}^{n+1} \text{ is a complex line} \} \]

Real blow up of \( \mathbb{CP}_n \) along \( \mathbb{RP}_n \)!

\[ F \supset \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \]

\[ \eta \]

\[ \mathbb{CP}_n \supset \mathbb{RP}_n \]

\( F \) inherits an involutive structure

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The X-ray transform on $\mathbb{CP}_n$ is well-understood.

Pullback of tensors under $\mu$ is well-understood.

Suitable global techniques on $\mathbb{CP}_n$ are available:

- $n = 2$: Penrose transform of $H^1(F_{1,2}(\mathbb{C}^2), \Theta) = 0$ etc.
- $n \geq 3$: BGG-like: $0 \to \Lambda^1 \to \bigotimes^2 \Lambda^1 \to \bigoplus \Lambda^1$ etc.