Symmetries via Lie algebra cohomology

Michael Eastwood

Australian National University

The Levi-Civita connection

Given g_{ab} a metric, $\exists!$ connection ∇_a characterised by

- ∇_a is torsion-free
- $\nabla_a g_{bc} = 0$.

Proof Choose any D_a torsion-free. Consider

$$\nabla_a \phi_b = D_a \phi_b - \Gamma_{ab}{}^c \phi_c.$$

Want: $\boxed{\Gamma_{abc} = \Gamma_{(ab)c}}$ and $\boxed{\Gamma_{a(bc)} = \frac{1}{2} D_a g_{bc}}$ but

$$\begin{array}{ccc} \Lambda^1 \otimes \Lambda^2 & \xrightarrow{\cong} & \Lambda^2 \otimes \Lambda^1 \\ K_{abc} & \longmapsto & K_{[ab]c} \\ \parallel & & \\ K_{a[bc]} & & \end{array}$$

NB

QED!

Riemannian symmetries

A vector field X^a is a Killing field if and only if

$$\mathcal{L}_X g_{bc} = 0.$$

But, for any torsion-free connection ∇_a,

$$\mathcal{L}_X \phi_b = X^a \nabla_a \phi_b + \phi_a \nabla_b X^a$$

so, if we use the Levi-Civita connection, then

$$\begin{aligned} \mathcal{L}_X g_{bc} &= X^a \nabla_a g_{bc} + g_{ac} \nabla_b X^a + g_{ba} \nabla_c X^a \\ &= \nabla_b X_c + \nabla_c X_b. \end{aligned}$$

Thus, X^a is a Killing field $\iff$ $\boxed{\nabla_{(a} X_{b)} = 0}$.

Prolongation of Killing equation

$\nabla_{(a}X_{b)} = 0 \iff \nabla_a X_b = K_{ab}$ for $K_{ab} = K_{[ab]}$.

But then,
$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= R_{bc}{}^d{}_a X_d. \end{aligned}$$

Therefore, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{aligned} \nabla_a X_b &= K_{ab} \\ \nabla_a K_{bc} &= R_{bc}{}^d{}_a X_d \end{aligned}}$

Hence, Killing fields $\leftrightarrow$ covariant constant sections of $V \equiv \Lambda^1 \oplus \Lambda^2$ with connection

$$\begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} \xmapsto{\nabla_a} \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} - R_{bc}{}^d{}_a X_d \end{bmatrix}.$$

Coupled de Rham sequence

$$\begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} \xrightarrow{\nabla} \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} - R_{bc}{}^d{}_a X_d \end{bmatrix}$$

$$\begin{array}{ccc} \Lambda^1 & \longrightarrow & \Lambda^1 \otimes \Lambda^1 \\ \oplus & \times & \oplus \\ \Lambda^2 & \longrightarrow & \Lambda^1 \otimes \Lambda^2 \end{array}$$

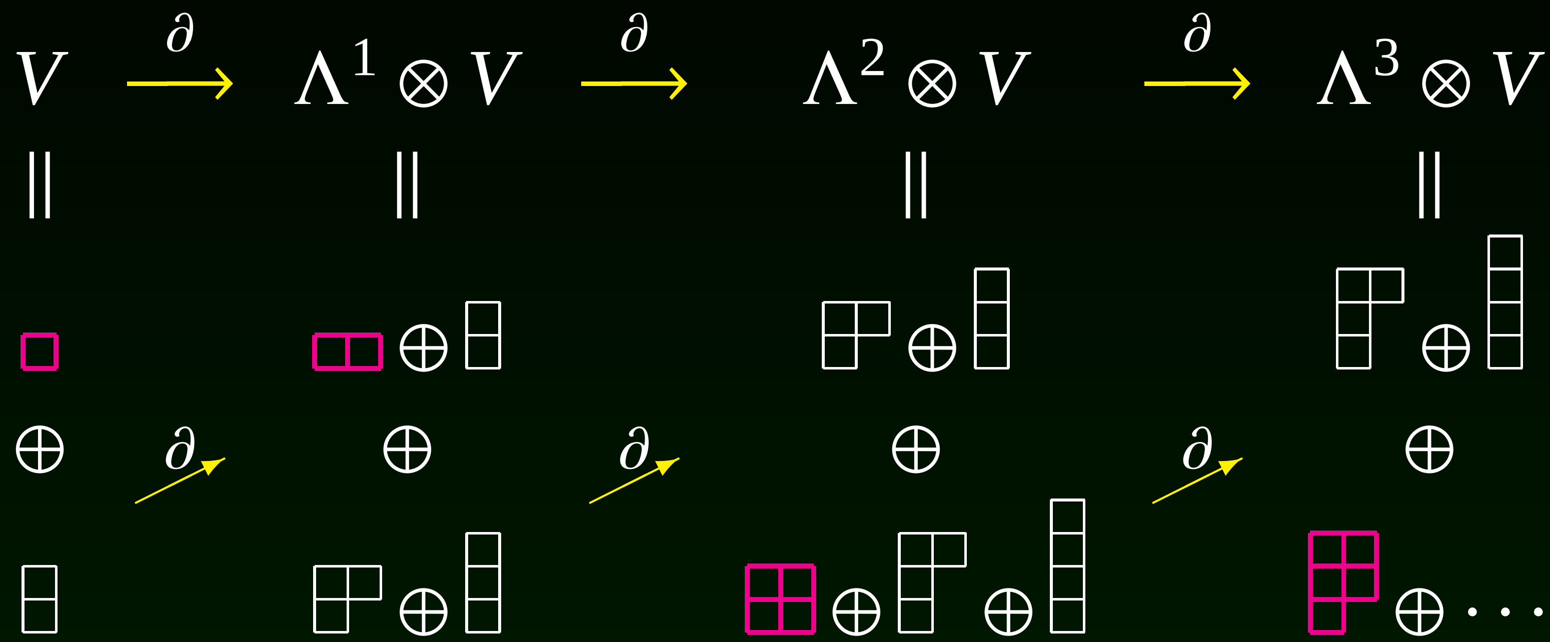
$$\begin{array}{ccccccccc} V & \xrightarrow{\nabla} & \Lambda^1 \otimes V & \xrightarrow{\nabla} & \Lambda^2 \otimes V & \xrightarrow{\nabla} & \Lambda^3 \otimes V & \xrightarrow{\nabla} & \cdots \\ \| & & \| & & \| & & \| & & \\ \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 & & \cdots \\ \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \\ \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2 & & \cdots \end{array}$$

NB

$$\Lambda^p \otimes \Lambda^2 \ni K_{a\cdots bcd} \xrightarrow{\partial} K_{[a\cdots bc]d} \in \Lambda^{p+1} \otimes \Lambda^1$$

$$K_{a\cdots bcd} = K_{[a\cdots b][cd]}$$

Decompose into irreducibles



Lie algebra cohomology! (Kostant 1961)

Suspend disbelief

EG: $\ker : \Lambda^2 \otimes \Lambda^2 \xrightarrow{\partial} \Lambda^3 \otimes \Lambda^1$

$= \{K_{abcd} = K_{[ab][cd]} \text{ s.t. } K_{[abc]d} = 0\}$

$= \{$Riemann curvature tensors$\}$!

Curvature

$$\begin{array}{ccccc} V & \xrightarrow{\nabla} & \Lambda^1 \otimes V & \xrightarrow{\nabla} & \Lambda^2 \otimes V \\ \| & & \| & & \| \\ \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 \\ \oplus & & \oplus & & \oplus \\ \Lambda^2 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^2 \end{array}$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \end{array} \right] = \left[\begin{array}{c} 0 \leftsquigarrow \text{by design} \\ R \bowtie K + (\nabla R) \bowtie X \end{array} \right]$$

$R \bowtie K$: $2R_{ab}{}^e{}_{[c}K_{d]e} + 2R_{cd}{}^e{}_{[a}K_{b]e}$

$(\nabla R) \bowtie X$: $(\nabla_b R_{cd}{}^e{}_a)X_e - (\nabla_a R_{cd}{}^e{}_b)X_e$

Flat $\iff R_{abcd} = \lambda(g_{ac}g_{bd} - g_{bc}g_{ad})$

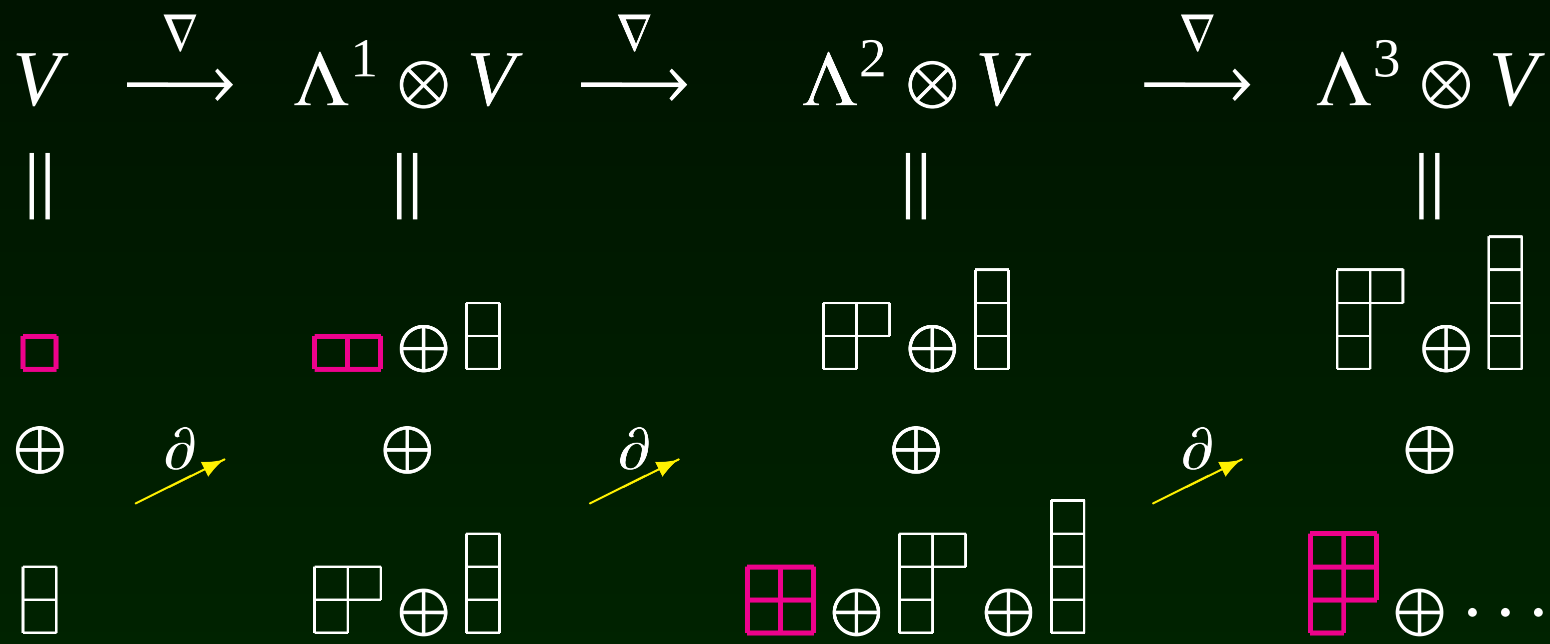
$\iff$ constant curvature.

Maximal symmetry

Spherical	$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$	$\mathrm{SO}(n+1)$
Euclidean	$R_{abcd} = 0$	$\mathrm{SO}(n) \ltimes \mathbb{R}^n$
Hyperbolic	$R_{abcd} = -g_{ac}g_{bd} + g_{bc}g_{ad}$	$\mathrm{SO}(n,1)$

$$\dim = \dim \Lambda^1 + \dim \Lambda^2 = n + n(n-1)/2 = n(n+1)/2$$

Diagram chasing in the constant curvature case...

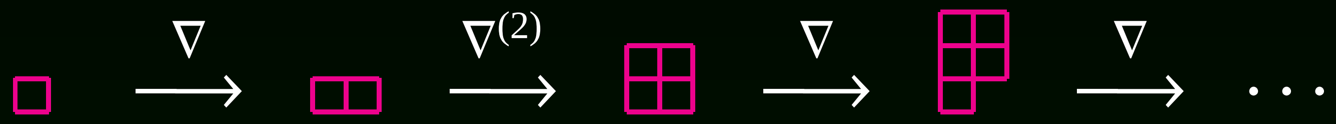


leads to...

BGG resolutions

The locally exact complexes

Riemannian deformation



$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots$$

de Rham

are Bernstein-Gelfand-Gelfand resolutions.

$$n\text{-sphere} = \mathrm{SL}(n+1,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} \lambda & * \cdots * \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix}, \lambda > 0 \right\} = G/P,$$

where G is semisimple and P is parabolic.
The round n-sphere is projectively flat.

Lie algebra cohomology

$\mathfrak{u}$ = Lie algebra $\qquad$ $\mathbb{V}$ = $\mathfrak{u}$-module

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{u}, \mathbb{V}) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2\mathfrak{u}, \mathbb{V}) \xrightarrow{\partial} \cdots$$

$$\partial v(X) = Xv \qquad \partial\phi(X \wedge Y) = \phi([X,Y]) - X\phi(Y) + Y\phi(X) \qquad \cdots$$

$$\rightsquigarrow \boxed{H^p(\mathfrak{u}, \mathbb{V})}$$

$$\mathfrak{sl}(n+1, \mathbb{R}) \ni \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array}\right]$$

$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Let $\mathfrak{u} = \mathfrak{g}_{-1}$ $(\Rightarrow \mathfrak{u}^* = \mathfrak{g}_1)$

Let $\mathbb{V} = \Lambda^2\mathbb{R}^{n+1}|_{\mathfrak{g}_{-1}}$

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2\mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial}$$

Geometric import

Kostant's Bott-Borel-Weil Theorem $\Longrightarrow$

$H^p(\mathfrak{g}_{-1}, \mathbb{V}) = \square, \square\!\square, \boxplus, \cdots$ as $\mathrm{SL}(n, \mathbb{R})$-modules

$$
\begin{array}{ccccccccccc}
0 & \rightarrow & \mathbb{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^2\mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^3\mathfrak{g}_1 \otimes \mathbb{V} \\
 & & \wr & & \wr & & \wr & & \wr \\
0 & \rightarrow & V & \xrightarrow{\partial} & \Lambda^1 \otimes V & \xrightarrow{\partial} & \Lambda^2 \otimes V & \xrightarrow{\partial} & \Lambda^3 \otimes V \\
 & & \| & & \| & & \| & & \| \\
 & & \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 \\
 & & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus \\
 & & \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2
\end{array}
$$

Previously suspected tensor identities are justified !!

Application to Killing operator

Range of the Killing operator? Given $\omega_{ab} = \omega_{(ab)}$,

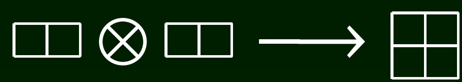
$$\omega_{ab} = \nabla_{(a}X_{b)} \quad \Leftrightarrow \quad \begin{bmatrix} \omega_{ab} \\ 2\nabla_{[b}\omega_{c]a} \end{bmatrix} = \nabla_a \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix}.$$

EG, on S^n (either locally or globally $(H^1(S^n, \mathbb{R}) = 0)$)

$$\nabla_a \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab}X_c - g_{ac}X_b \end{bmatrix}$$

$$\omega_{ab} = \nabla_{(a}X_{b)} \Leftrightarrow \boxed{\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0}, \text{ where}$$

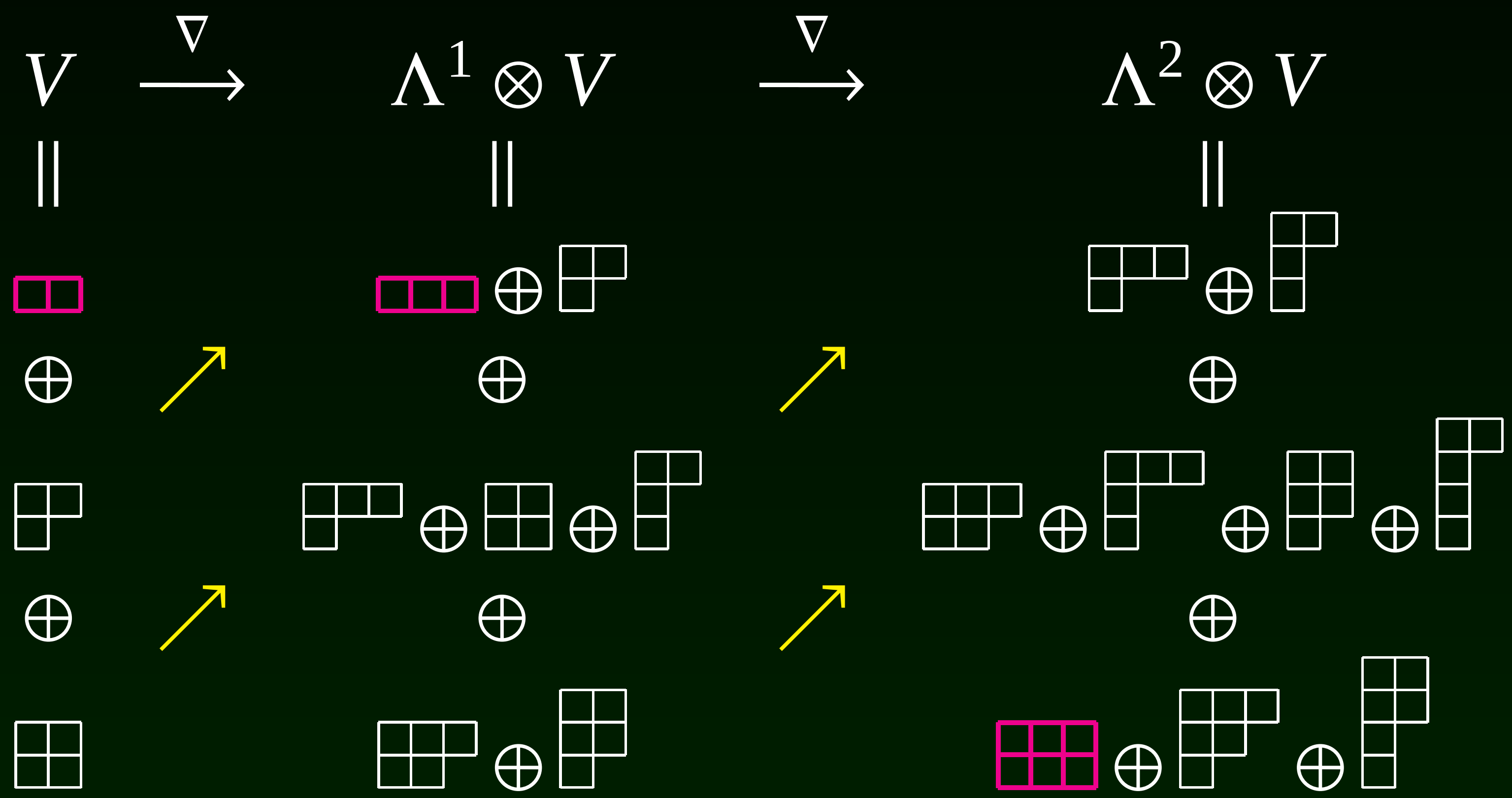
$$\pi(\Phi_{acbd} = \Phi_{[ab][cd]})$$



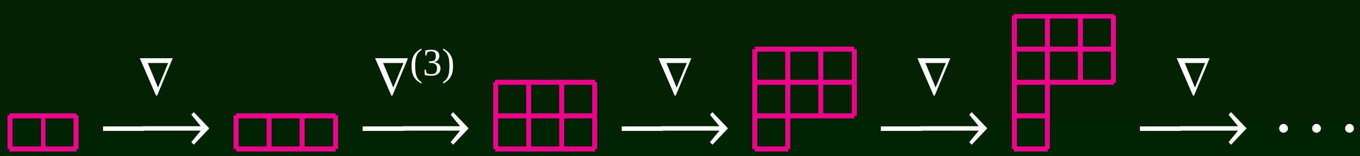
Higher Killing operators

prolong using Lie algebra cohomology

$$X_{bc} = X_{(bc)} \mapsto \nabla_{(a}X_{bc)} \rightsquigarrow$$



BGG



Killing operator on CP_n

Connection on $\Lambda^1 \oplus \Lambda^2$:–

$$\left[\begin{array}{l} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b - J_{ab} J_c{}^d X_d + J_{ac} J_b{}^d X_d + 2 J_{bc} J_a{}^d X_d \end{array}\right]$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \end{array}\right] = 4 \left[\begin{array}{c} 0 \\ J_{c[a} \tilde{K}_{b]d} - J_{d[a} \tilde{K}_{b]c} - J_{ab} \tilde{K}_{cd} - J_{cd} \tilde{K}_{ab} \end{array}\right]$$

where $\tilde{K}_{ab} \equiv J_{[a}{}^e K_{b]e}$. Connection on $\Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$:–

$$\left[\begin{array}{c} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b + J_{ab} L_c - J_{ac} L_b - J_{bc} L_a + J_{bc} J_a{}^d X_d \\ \nabla_a L_b + J_a{}^d K_{bd} \end{array}\right]$$

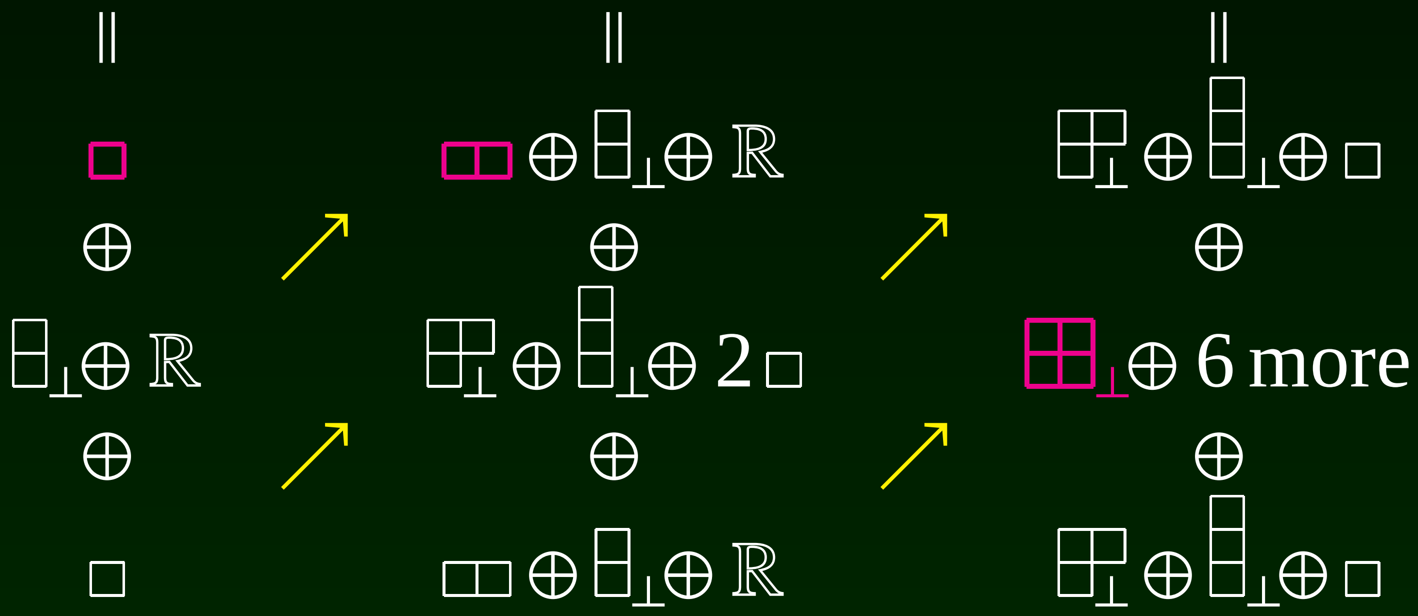
Killing operator on $\mathbf{CP_n}$ cont'd

Curvature

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix} X_c \\ K_{cd} \\ L_c \end{bmatrix} = 2J_{ab}\begin{bmatrix} L_c + J_c{}^e X_e \\ -J_c{}^e K_{de} + J_d{}^e K_{ce} \\ -X_c + J_c{}^e L_e \end{bmatrix}$$

Lie algebra cohomology (Heisenberg algebra)

$$V \xrightarrow{\nabla} \Lambda^1 \otimes V \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes V$$



Killing operator on $\mathrm{CP_n}$ cont'd

- Kernel of Killing $\Gamma(\mathbb{CP}_n, \square) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \square\!\square)$

$$\boxed{\begin{aligned} L_c &= -J_c{}^e X_e \\ J_{[c}{}^e K_{d]e} &= 0 \end{aligned}} \qquad \mathfrak{su}(n+1) \subset \mathfrak{sp}(2(n+1), \mathbb{R})$$

- Range of Killing $\Gamma(\mathbb{CP}_n, \square) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \square\!\square)$

$$\omega_{ab} = \nabla_{(a} X_{b)} \Leftrightarrow \boxed{(\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}))_\perp = 0}$$

Further reading

- D.M.J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, Jour. Reine Angew. Math. 537 (2001) 67–103.
- A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, Ann. Math. 154 (2001) 97–113.
- T.P. Branson, A. Čap, M.G. Eastwood, and A.R. Gover, Prolongations of geometric overdetermined systems, Internat. Jour. Math. 17 (2006) 641–664.
- A. Čap and J. Slovák, Parabolic Geometries 1, Surveys vol. 154, AMS 2009.

THANK YOU

THE END