Isolated hypersurface singularities, associated forms, and their invariants

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[on the work of Alexander Isaev et al.]

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Some references

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Prototype

Saito's simple elliptic singularities $\tilde{E}_6$ (cf. Chen–Seeley–Yau)

$$\mathbb{C}^3 \supset V_t = \{x^3 + y^3 + z^3 + txyz = 0\}, \qquad t^3 + 27 \neq 0$$

Observations

- It's homogeneous
- It's a cone on the elliptic curve

$$E_t = \{[x, y, z] \in \mathbb{CP}_2 \mid x^3 + y^3 + z^3 + txyz = 0\}$$

Consequence by means of blow-up

$$j = -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3}$$ is an invariant

Prototype cont'd

$$x^3 + y^3 + z^3 + txyz = 0 \quad \rightsquigarrow \quad -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3}$$

Q Is this all? NB change of coördinates

- $\omega^3 = 1 \quad x \mapsto \omega x \quad y \mapsto y \quad z \mapsto z \quad \rightsquigarrow \quad V_t \mapsto V_{\omega t}$
- $$\begin{array}{rcl} x & \mapsto & x + y + z \\ y & \mapsto & x + \omega y + \omega^2 z \\ z & \mapsto & x + \omega^2 y + \omega z \end{array} \quad \rightsquigarrow \quad V_t \mapsto V_{\frac{3(6-t)}{t+3}}$$

A Yes! The invariant j is complete for V_t.

Q Direct route to j? A Yes via the Milnor algebra!

Milnor algebras

$\mathcal{O}$ = germs of holomorphic functions at the origin in $\mathbb{C}^n$

$\mathcal{V}$ = germ of analytic hypersurface through $0 \in \mathbb{C}^n$

$\mathcal{V} \rightsquigarrow \mathcal{I} \subset \mathcal{O} \rightsquigarrow f \in \mathcal{O}$ s.t. $\mathcal{I} = \langle f \rangle \rightsquigarrow \mathcal{V} = \{f = 0\}$

$\mathcal{V} \rightsquigarrow \mathcal{A} = \mathcal{O}/\langle f, \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle$ moduli algebra

- $\mathcal{A}$ does not depend on choice of f nor coördinates
- $\mathcal{A} = 0 \iff \mathcal{V}$ is non-singular
- $\mathcal{A} = \mathbb{C} \iff \mathcal{V} \cong \{z_1^2 + \cdots + z_n^2 = 0\}$
- $0 < \dim \mathcal{A} < \infty \iff 0 \in \mathcal{V}$ is an isolated singularity

$\mathcal{V} \rightsquigarrow \mathcal{O}/\langle \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle$ Milnor algebra

Thm (Saito) Milnor = moduli $\iff \mathcal{V}$ is quasi-homogeneous

Back to prototype

$$V_t = \{x^3 + y^3 + z^3 + txyz = 0\}$$ homogeneous ✓

$\rightsquigarrow$

$$A_t = \mathbb{C}\langle 1, x, y, z, yz, zx, xy, xyz\rangle \text{ s.t. } x^2 = -\tfrac{t}{3}yz,\; y^2 = -\tfrac{t}{3}zx\; z^2 = -\tfrac{t}{3}xy$$

Observations

- $\dim A_t = 8$
- maximal ideal $\mathfrak{m} = (x, y, z)$ is codimension 1
- $\mathfrak{m}/\mathfrak{m}^2 = \mathbb{C}\langle x, y, z\rangle$ is 3-dimensional
- $\mathfrak{m}^3/\mathfrak{m}^4 = \mathfrak{m}^3 = \mathbb{C}\langle xyz\rangle$ is 1-dimensional
- Multiplication $\bigodot^3 \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^3$ is invariantly defined

$\rightsquigarrow$ polynomial, invariantly defined up to scale (ternary cubic)

$$F_t(X, Y, Z) = tX^3 + tY^3 + tZ^3 - 18XYZ$$

Extracting invariants

In A_t we have e.g. $x^3 = x(x^2) = -\frac{t}{3}xyz$, but

$$x^2y = -\frac{t}{3}y^2z = \frac{t^2}{9}xz^2 = -\frac{t^3}{27}x^2y \quad \Longrightarrow \quad x^2y = 0.$$

$\rightsquigarrow$ $\boxed{F = t(X^1)^3 + t(X^2)^3 + t(X^3)^3 - 18X^1X^2X^3}$. Now write

$$F = \sum_{1\le a,b,c\le 3} F_{abc}X^aX^bX^c$$ for the symmetric tensor F_{abc},

pick a skew form ϵ^{abc}, and assemble the classical invariants

$$J = F_{abc}F_{def}F_{ghi}F_{jkl}\epsilon^{adg}\epsilon^{bej}\epsilon^{chk}\epsilon^{fil} = \text{constant} \times (t^3 + 27)$$

$$K = F_{abc}F_{def}F_{ghi}F_{jkl}F_{pqr}F_{stu}\epsilon^{aiu}\epsilon^{bdg}\epsilon^{cej}\epsilon^{fhp}\epsilon^{lrs}\epsilon^{kqt}$$

Extracting invariants cont'd

Note that

- J is homogeneous of degree 4 in ϵ^{abc}
- K is homogeneous of degree 6 in ϵ^{abc}

$\mathbb{C}[J, K]$

$\mathrm{SL}(3, \mathbb{C})$-invariants

Therefore

$$j = \frac{J^3 - 6K^2}{J^3}$$

is invariant under $\epsilon^{abc} \mapsto \lambda\epsilon^{abc}$ and thus an absolute invariant

Calculation (with a computer) yields

$$F = tX^3 + tY^3 + tZ^3 - 18XYZ \implies j = -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3},$$

as required.

Generalities

Recall $\mathcal{V} \longmapsto \mathcal{A}$, the moduli algebra.

Theorem (Mather–Yau 1982): conversely $\mathcal{A}$ as an algebra determines $\mathcal{V}$ up to biholomorphic equivalence.

Thus, we can extract all invariants of $\mathcal{V}$ from $\mathcal{A}$ (in principle).

Properites of $\mathcal{A}$: for $\mathcal{V}$ isolated quasi-homogeneous

- $0 < \dim \mathcal{A} < \infty$
- $\mathcal{A}$ is local, Artinian, ...
- $\mathcal{A}$ is Gorenstein: $\dim \mathrm{Ann}(\mathfrak{m}) = 1$
- $\mathcal{A}$ is graded: $\mathcal{A} = \bigoplus_{j \geq 0} \mathcal{L}_j$ s.t. $\mathcal{L}_0 \simeq \mathbb{C} \quad \mathcal{L}_j \mathcal{L}_k \subseteq \mathcal{L}_{j+k}$

Theorem (Fels–Isaev–Kaup–Kruzhilin) $\mathcal{A}$ graded Gorenstein. Nil-polynomials $P : \mathfrak{m}/\mathrm{Ann}(\mathfrak{m}) \to \mathbb{C}$. . . . . , P determine $\mathcal{A}$!

Prototype revisited

$$x^3 + y^3 + z^3 + txyz \rightsquigarrow tX^3 + tY^3 + tZ^3 - 18XYZ$$

$$j = -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3} \qquad j = -\frac{1728(t^3 + 27)^3}{t^3(t^3 - 216)^3}$$

Reciprocal !

Weierstraß

$$y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3 \rightsquigarrow$$

$$Y^2 Z - 4\left(\frac{{g_2}^3}{27{g_3}^2 - {g_2}^3}\right) X^3 + g_2 X Z^2 + g_3 Z^3$$

Binary sextics

Sylvester

$$ax^6 + by^6 + cz^6 + dxyz(x-y)(y-z)(z-x) \qquad x+y+z=0$$

Associated binary octavic

$$\cdots + \begin{pmatrix} 8600c^3d^5 + 3000c^4d^4 - 36160d^6c^2 \\ + 23400c^4d^3a + 151920c^3d^4a - 14624d^6ca \\ + 1512d^8 + 702120d^4c^2a^2 - 13040d^5a^2c \\ + 435456d^3c^3a^2 - 28720d^6a^2 + \cdots \end{pmatrix} XY^7 + \cdots$$

Nevertheless, the sextic is determined by the classical invariants of the associated octavic.

Binary quantics now done by Alper and Isaev!

Aside: other binary sextics

Cannot be thrown into Sylvester canonical form!

- x^4y^2
- $x^4(x^2+y^2)$
- x^3y^3
- x^5y
- $x(x^5+y^5)$
- $2x^6+18x^5y+10x^3y^3-y^6$
- $184x^6-192x^5y-300x^4y^2-320x^3y^3-150x^2y^4-48xy^5+23y^6$

Special points
in the moduli space of genus 2 Riemann surfaces!

THE END

THANK YOU