Isolated hypersurface singularities, associated forms, and their invariants

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[ on the work of Alexander Isaev et al. ]

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Some references


On the arXiv

- J. Alper and A.V. Isaev, Associated forms in classical invariant theory
- J. Alper and A.V. Isaev, Associated forms and hypersurface singularities: the binary case
- J. Alper, A.V. Isaev, and N.G. Kruzhilin, Associated forms of binary quartics and ternary cubics
- A.V. Isaev, A criterion for isomorphism of Artinian Gorenstein algebras
Saito’s simple elliptic singularities $\tilde{E}_6$ (cf. Chen–Seeley–Yau)

$$\mathbb{C}^3 \supset V_t = \{x^3 + y^3 + z^3 + txyz = 0\}, \quad t^3 + 27 \neq 0$$

**Observations**

- It’s homogeneous
- It’s a cone on the elliptic curve

$$E_t = \{[x, y, z] \in \mathbb{CP}_2 \mid x^3 + y^3 + z^3 + txyz = 0\}$$

**Consequence** by means of blow-up

$$j = -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3}$$

is an invariant
Prototype cont’d

\[ x^3 + y^3 + z^3 + txyz = 0 \quad \leadsto \quad -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3} \]

**Q** Is this all?  **NB** change of coördinates

- \[ \omega^3 = 1 \quad x \mapsto \omega x \quad y \mapsto y \quad z \mapsto z \quad \leadsto \quad V_t \mapsto V_{\omega t} \]
- \[ x \mapsto x + y + z \]
- \[ y \mapsto x + \omega y + \omega^2 z \quad \leadsto \quad V_t \mapsto V_{\frac{3(6-t)}{t+3}} \]
- \[ z \mapsto x + \omega^2 y + \omega z \]

**A** Yes! The invariant \( j \) is complete for \( V_t \).

**Q** Direct route to \( j \)?  **A** Yes via the Milnor algebra!
Milnor algebras

\[ \mathcal{O} = \text{germs of holomorphic functions at the origin in } \mathbb{C}^n \]
\[ \mathcal{V} = \text{germ of analytic hypersurface through } 0 \in \mathbb{C}^n \]
\[ \mathcal{V} \sim \mathcal{I} \subset \mathcal{O} \sim f \in \mathcal{O} \text{ s.t. } \mathcal{I} = \langle f \rangle \sim \mathcal{V} = \{ f = 0 \} \]
\[ \mathcal{V} \sim \mathcal{A} = \mathcal{O}/\langle f, \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle \text{ moduli algebra} \]

- \( \mathcal{A} \) does not depend on choice of \( f \) nor coordinates
- \( \mathcal{A} = 0 \iff \mathcal{V} \text{ is non-singular} \)
- \( \mathcal{A} = \mathbb{C} \iff \mathcal{V} \cong \{ z_1^2 + \cdots + z_n^2 = 0 \} \)
- \( 0 < \dim \mathcal{A} < \infty \iff 0 \in \mathcal{V} \text{ is an isolated singularity} \)

\[ \mathcal{V} \sim \mathcal{O}/\langle \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle \text{ Milnor algebra} \]

Thm (Saito) Milnor = moduli \iff \( \mathcal{V} \) is quasi-homogeneous
\[ V_t = \{ x^3 + y^3 + z^3 + txyz = 0 \} \quad \text{homogeneous} \checkmark \]

\[ A_t = \mathbb{C}\langle 1, x, y, z, yz, zx, xy, xyz \rangle \quad \text{s.t.} \quad x^2 = -\frac{t}{3}yz, \quad y^2 = -\frac{t}{3}zx, \quad z^2 = -\frac{t}{3}xy \]

Observations

- \( \dim A_t = 8 \)
- maximal ideal \( m = (x, y, z) \) is codimension \( 1 \)
- \( m/m^2 = \mathbb{C}\langle x, y, z \rangle \) is \( 3 \)-dimensional
- \( m^3/m^4 = m^3 = \mathbb{C}\langle xyz \rangle \) is \( 1 \)-dimensional
- Multiplication \( \otimes^3 m/m^2 \rightarrow m^3 \) is invariants defined

polynomial, invariants defined up to scale (ternary cubic)

\[ F_t(X, Y, Z) = tx^3 + ty^3 + tz^3 - 18XYZ \]
Extracting invariants

In $A_t$ we have e.g. $x^3 = x(x^2) = -\frac{t}{3}xyz$, but

$$x^2y = -\frac{t}{3}y^2z = \frac{t^2}{9}xz^2 = -\frac{t^3}{27}x^2y \implies x^2y = 0.$$ 

$$\text{Now write } F = t(X^1)^3 + t(X^2)^3 + t(X^3)^3 - 18X^1X^2X^3.$$ 

$$F = \sum_{1 \leq a,b,c \leq 3} F_{abc}X^aX^bX^c$$ for the symmetric tensor $F_{abc}$, pick a skew form $\epsilon^{abc}$, and assemble the classical invariants

$$J = F_{abc}F_{def}F_{ghi}F_{jkl}\epsilon^{adg}\epsilon^{bej}\epsilon^{chk}\epsilon^{fil} = \text{constant} \times (t^3 + 27)$$

$$K = F_{abc}F_{def}F_{ghi}F_{jkl}F_{pqr}F_{stu}\epsilon^{aiu}\epsilon^{bdg}\epsilon^{cej}\epsilon^{fhp}\epsilon^{lrs}\epsilon^{kqt}$$
Note that

- $J$ is homogeneous of degree 4 in $\epsilon^{abc}$
- $K$ is homogeneous of degree 6 in $\epsilon^{abc}$

Therefore

$$j = \frac{J^3 - 6K^2}{J^3}$$

is invariant under $\epsilon^{abc} \mapsto \lambda \epsilon^{abc}$ and thus an absolute invariant.

Calculation (with a computer) yields

$$F = tX^3 + tY^3 + tZ^3 - 18XYZ \implies j = -\frac{t^3(t^3 - 216)^3}{1728(t^3 + 27)^3},$$

as required.
Generalities

Recall $\mathcal{V} \leftrightarrow \mathcal{A}$, the moduli algebra.

Theorem (Mather–Yau 1982): conversely $\mathcal{A}$ as an algebra determines $\mathcal{V}$ up to biholomorphic equivalence.

Thus, we can extract all invariants of $\mathcal{V}$ from $\mathcal{A}$ (in principle).

Properties of $\mathcal{A}$: for $\mathcal{V}$ isolated quasi-homogeneous

- $0 < \dim \mathcal{A} < \infty$
- $\mathcal{A}$ is local, Artinian, ...
- $\mathcal{A}$ is Gorenstein: $\dim \text{Ann}(m) = 1$
- $\mathcal{A}$ is graded: $\mathcal{A} = \bigoplus_{j \geq 0} \mathcal{L}_j \text{ s.t. } \mathcal{L}_0 \cong \mathbb{C} \quad \mathcal{L}_j \mathcal{L}_k \subseteq \mathcal{L}_{j+k}$

Theorem (Fels–Isaev–Kaup–Kruzhilin) $\mathcal{A}$ graded Gorenstein. Nil-polynomials $P : m/\text{Ann}(m) \to \mathbb{C}$. ........, $P$ determine $\mathcal{A}$!
Prototype revisited

\[ x^3 + y^3 + z^3 + txyz \quad \mapsto \quad tX^3 + tY^3 + tZ^3 - 18XYZ \]

\[ j = \frac{-t^3(t^3 - 216)^3}{1728(t^3 + 27)^3} \quad \Downarrow \quad \Downarrow \quad \Downarrow \]

\[ j = \frac{-1728(t^3 + 27)^3}{t^3(t^3 - 216)^3} \]

Weierstraß

\[ y^2z - 4x^3 + g_2xz^2 + g_3z^3 \quad \mapsto \quad Y^2Z - 4 \left( \frac{g_2^3}{27g_3^2 - g_2^3} \right) X^3 + g_2XZ^2 + g_3Z^3 \]
Binary sextics

Sylvester

\[ ax^6 + by^6 + cz^6 + dxyz(x - y)(y - z)(z - x) \quad x + y + z = 0 \]

Associated binary octavic

\[
\begin{align*}
&+ 8600c^3d^5 + 3000c^4d^4 - 36160d^6c^2 \\
&\quad + 23400c^4d^3a + 151920c^3d^4a - 14624d^6ca \\
&\quad + 1512d^8 + 702120d^4c^2a^2 - 13040d^5a^2c \\
&\quad + 435456d^3c^3a^2 - 28720d^6a^2 + \cdots \\
\end{align*}
\]

Nevertheless, the sextic is determined by the classical invariants of the associated octavic.

Binary quantics now done by Alper and Isaev!
Aside: other binary sextics

**Cannot be thrown into Sylvester canonical form!**

- $x^4y^2$
- $x^4(x^2 + y^2)$
- $x^3y^3$
- $x^5y$
- $x(x^5 + y^5)$
- $2x^6 + 18x^5y + 10x^3y^3 - y^6$
- $184x^6 - 192x^5y - 300x^4y^2 - 320x^3y^3 - 150x^2y^4 - 48xy^5 + 23y^6$

**Special points**
in the moduli space of genus 2 Riemann surfaces!
THE END

THANK YOU