Background on c-projective geometry

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[following the work of others]

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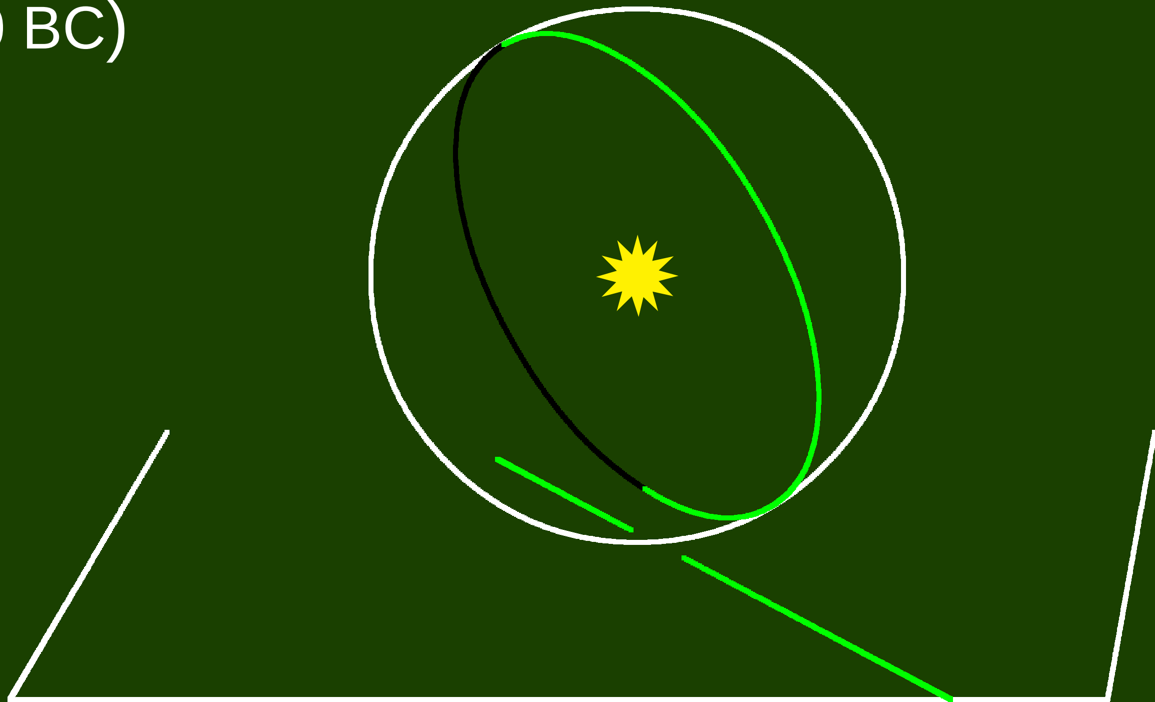
Classical question

Given a Riemannian metric, is there another with the same geodesics as unparameterised curves?

EG (Thales circa 600 BC)
the round sphere



$$\frac{(1+y^2)\,dx^2 - xy\,dx\,dy + (1+x^2)\,dy^2}{(1+x^2+y^2)^2}$$

Better (classical) question

Given a projective structure, is there a metric connection in the projective class and, if so, how many?

Geometric Def$^{\underline{n}}$ Two torsion-free affine connections ∇_α and $\hat{\nabla}_\alpha$ are projectively equivalent iff they have the same geodesics as unparameterised curves

Operational Def$^{\underline{n}}$ Two torsion-free affine connections ∇_α and $\hat{\nabla}_\alpha$ are projectively equivalent iff

$$\hat{\nabla}_\alpha \phi_\beta = \nabla_\alpha \phi_\beta - \Upsilon_\alpha \phi_\beta - \Upsilon_\beta \phi_\alpha$$

for some 1-form Υ_α

$\lightning\lightning\lightning$ better derivation $\lightning\lightning\lightning$

Tensors on oriented smooth manifolds

Frame bundle $\leftrightsquigarrow$ principal $\mathrm{GL}_+(n,\mathbb{R})$-bundle

Tangent bundle $\leftrightsquigarrow$ $\mathrm{GL}_+(n,\mathbb{R})$ acting on $\mathbb{R}^n$ (defining rep$^{\underline{\mathrm{n}}}$)
Cotangent bundle $\leftrightsquigarrow$ $\mathrm{GL}_+(n,\mathbb{R})$ acting on $(\mathbb{R}^n)^*$
Symmetric covariant tensors $\leftrightsquigarrow$ $\mathrm{GL}_+(n,\mathbb{R})$ acting on $\bigodot^2(\mathbb{R}^n)^*$

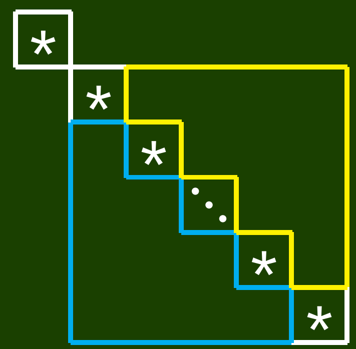
Embed $\quad \mathrm{GL}_+(n,\mathbb{R}) \hookrightarrow \mathrm{SL}(n+1,\mathbb{R})$

$$\lambda^{-1}B \longleftarrow \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) \quad \lambda > 0$$

$$\mathfrak{gl}(n,\mathbb{R}) \ni \left(\begin{array}{c|ccccc} * & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \end{array}\right)$$

(trace-free)

raising lowering Cartan



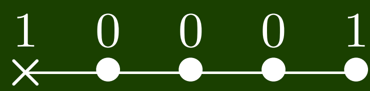
The (co-)tangent bundle

$$\mathfrak{gl}(n,\mathbb{R}) \ni \left(\begin{array}{c|ccccc} * & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \end{array}\right)$$

lowest weight vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

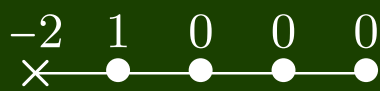
$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

minus lowest weight



$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

dual bundle

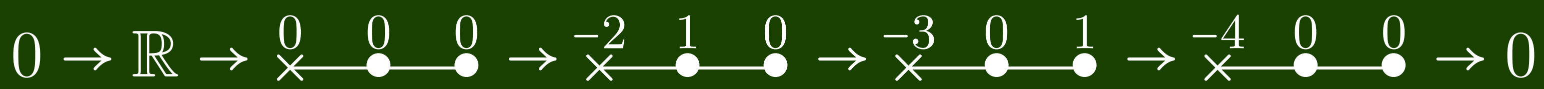
$\Lambda^1 =$ 

The de Rham complex

In three dimensions (grad, curl, div)

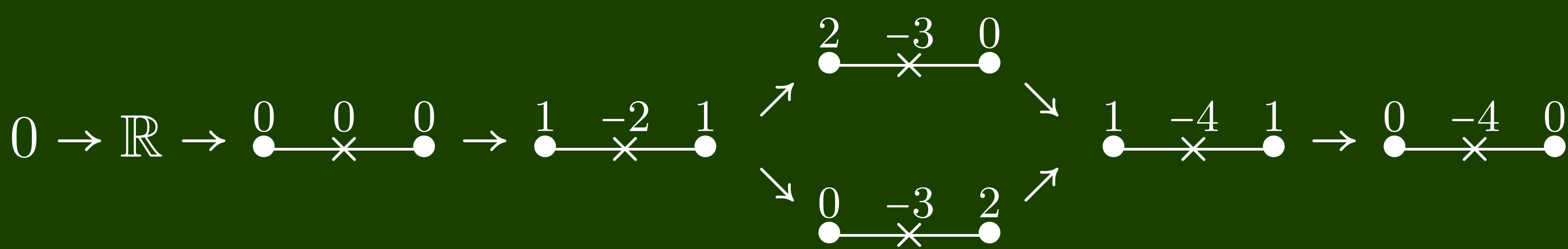
$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \to 0$$

Use our new notation!



Already redolent of projective differential geometry!

Cf. de Rham complex in four conformal dimensions



Easy algorithms!

Real projective geometry

Change of connection (contorsion)

$$\hat{\nabla}_\alpha \phi_\beta = \nabla_\alpha \phi_\beta - \Gamma_{\alpha\beta}{}^\gamma \phi_\gamma \qquad \text{for } \Gamma_{\alpha\beta}{}^\gamma \in \Lambda^1 \otimes \Lambda^1 \otimes T$$

$\rightsquigarrow$ change of torsion $\tau_{\alpha\beta}{}^\gamma = 2\Gamma_{[\alpha\beta]}{}^\gamma \in \Lambda^2 \otimes T$ $\rightsquigarrow$

$$\partial : \Lambda^1 \otimes \Lambda^1 \otimes T \to \Lambda^2 \otimes T$$

- $\operatorname{coker} \partial = 0$
- $\ker \partial = \bigodot^2 \Lambda^1 \otimes T$

$$= \overset{-4}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \otimes \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$$

$$= \overset{-3}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet} \oplus \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$$

$$\hat{\nabla}_\alpha \phi_\beta = \nabla_\alpha \phi_\beta - \Upsilon_\alpha \phi_\beta - \Upsilon_\beta \phi_\alpha$$

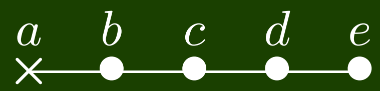
Projective differential geometry from thin air!

The flat model G/P

$$\mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R}) \Big/ \left\{ \left(\begin{array}{c|ccccc} * & * & * & \cdots & * & * \\ \hline 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \end{array} \right) \right\}$$

or (better)

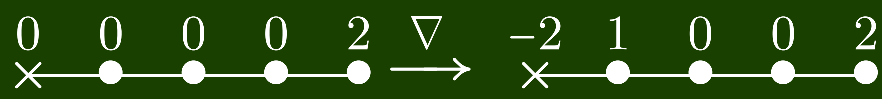
$$S^n = \mathrm{SL}(n+1,\mathbb{R}) \Big/ \left\{ \left(\begin{array}{c|ccccc} \lambda & * & * & \cdots & * & * \\ \hline 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \end{array} \right) \text{ for } \lambda > 0 \right\}$$

Irreducible homogeneous vector bundles 

PILDOs

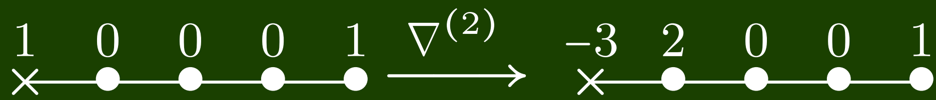
Examples

Metrisability



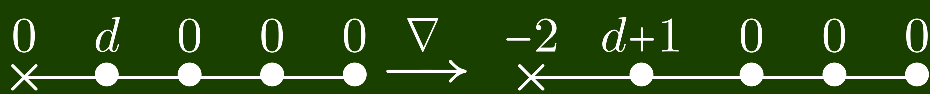
$$\odot^2 T(-2) \ni \sigma^{\beta\gamma} \longmapsto (\nabla_\alpha \sigma^{\beta\gamma})_\circ$$

Projective Killing



$$X^\gamma \longmapsto (\nabla_{(\alpha}\nabla_{\beta)}X^\gamma + \mathrm{P}_{\alpha\beta}X^\gamma + W_{\delta(\alpha}{}^{\gamma}{}_{\beta)}X^\delta)_\circ$$

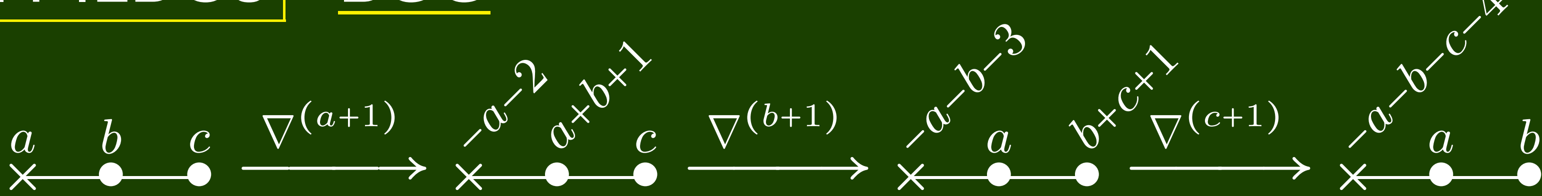
Killing tensors !!



$$\odot^d\Lambda^1(2d) \ni X_{\beta\gamma\cdots\delta} \longmapsto \nabla_{(\alpha}X_{\beta\gamma\cdots\delta)}$$

All PILDOs

BGG



Holomorphic projective geometry

Repeat previous discussion in the holomorphic category!

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

for some holomorphic 1-form Υ_a

Crucial difference! ☠☠! $\mathrm{GL}(n, \mathbb{C}) \not\to \mathrm{SL}(n+1, \mathbb{C})$

$$\lambda^{-1} B \longleftarrow \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

An $(n+1)$-fold covering

Do not be afraid: cf. spinors

Upshot: insist on an $(n+1)^{\mathrm{st}}$ root of the canonical bundle.

$\Omega^n = \mathcal{O}(-n-1)$ as on $\mathbb{CP}_n$ (the flat model)

Tensors on almost complex manifolds

Reduced structure group $\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n,\mathbb{R})$ by

$$\mathrm{GL}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(2n,\mathbb{R}) \text{ s.t. } M\mathbb{J} = \mathbb{J}M\},$$

where, for example, $\mathbb{J}$ is the $2n \times 2n$ matrix

$$\left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right).$$

Complexify

$$\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \cong \mathrm{GL}(n,\mathbb{C})^{\mathbb{C}} \hookrightarrow \mathrm{GL}(2n,\mathbb{C})$$

$$\mathbb{C}^{2n} \leftrightarrow \mathbb{C}T = T^{1,0} \oplus T^{0,1} = \{v \text{ s.t. } Jv = iv\} \oplus \{v \text{ s.t. } Jv = -iv\}.$$

Covering

 subgroup of $\mathrm{SL}(n+1,\mathbb{C})$

$$\left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

Homomorphism  $\to \mathrm{GL}(n,\mathbb{C})$ given by

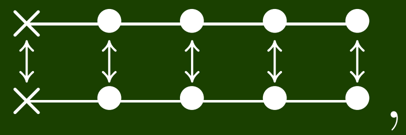
$$\left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) \longmapsto \lambda^{-1}B.$$

Recall: it is an $(n+1)$-fold covering.

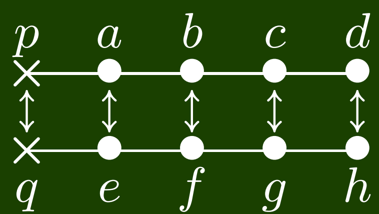
Representations

$$\mathrm{GL}(n,\mathbb{C})^{\mathbb{C}} \cong \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$$

Lie algebra (case $n = 5$)



and decorate



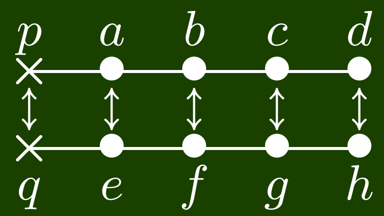
and optionally insist that

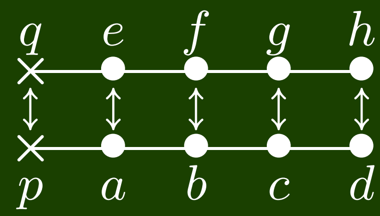
$$p + 2a + 3b + 4c + 5d \equiv q + 2e + 3f + 4g + 5h \bmod 6.$$

External tensor product

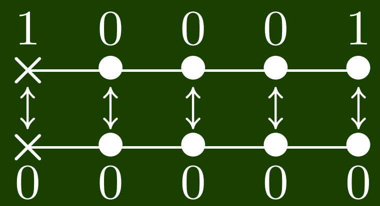
but keeps track of real structure.

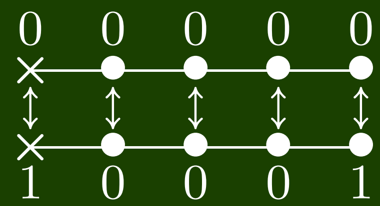
Conjugate bundles

Conjugate of 

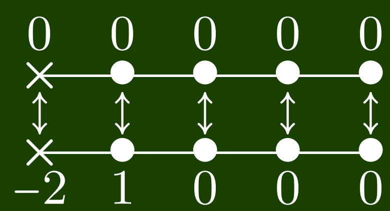
is 

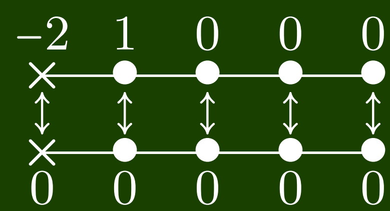
Prototype

$\mathbb{C}T = T^{1,0} \oplus T^{0,1} =$ 

$\oplus$ 

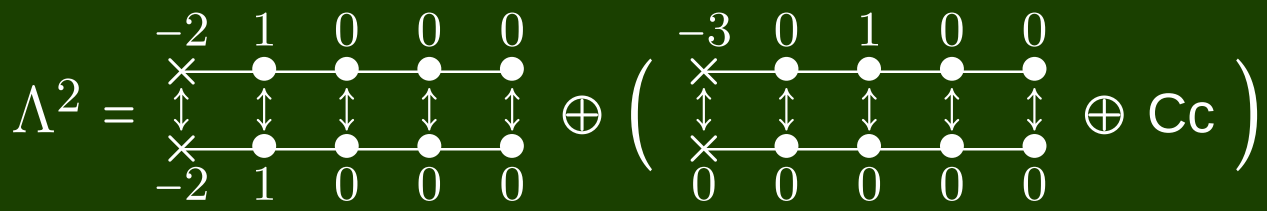
Dual

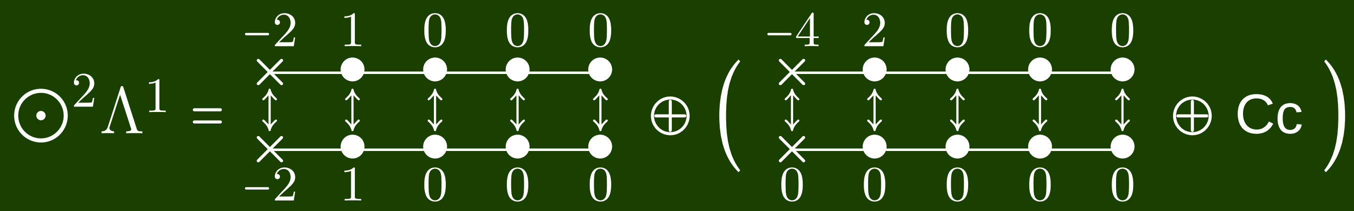
$\Lambda^1 = \Lambda^{0,1} \oplus \Lambda^{1,0} =$ 

$\oplus$ 

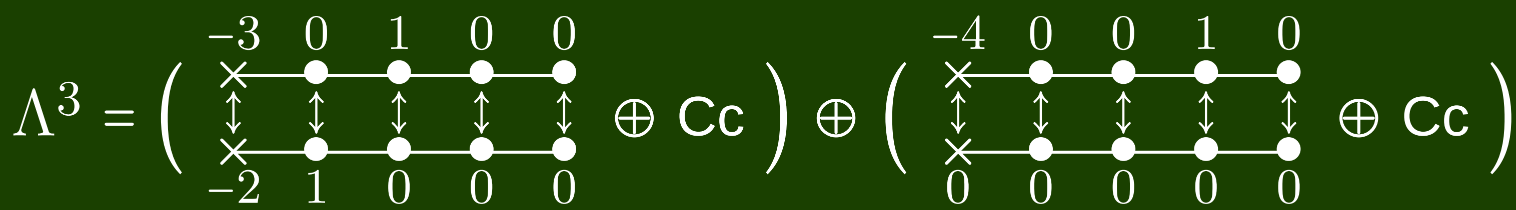
2-tensors and 3-forms

$$\Lambda^2 = \Lambda^{1,1} \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2})$$





$$\Lambda^3 = (\Lambda^{2,1} \oplus \Lambda^{1,2}) \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2})$$

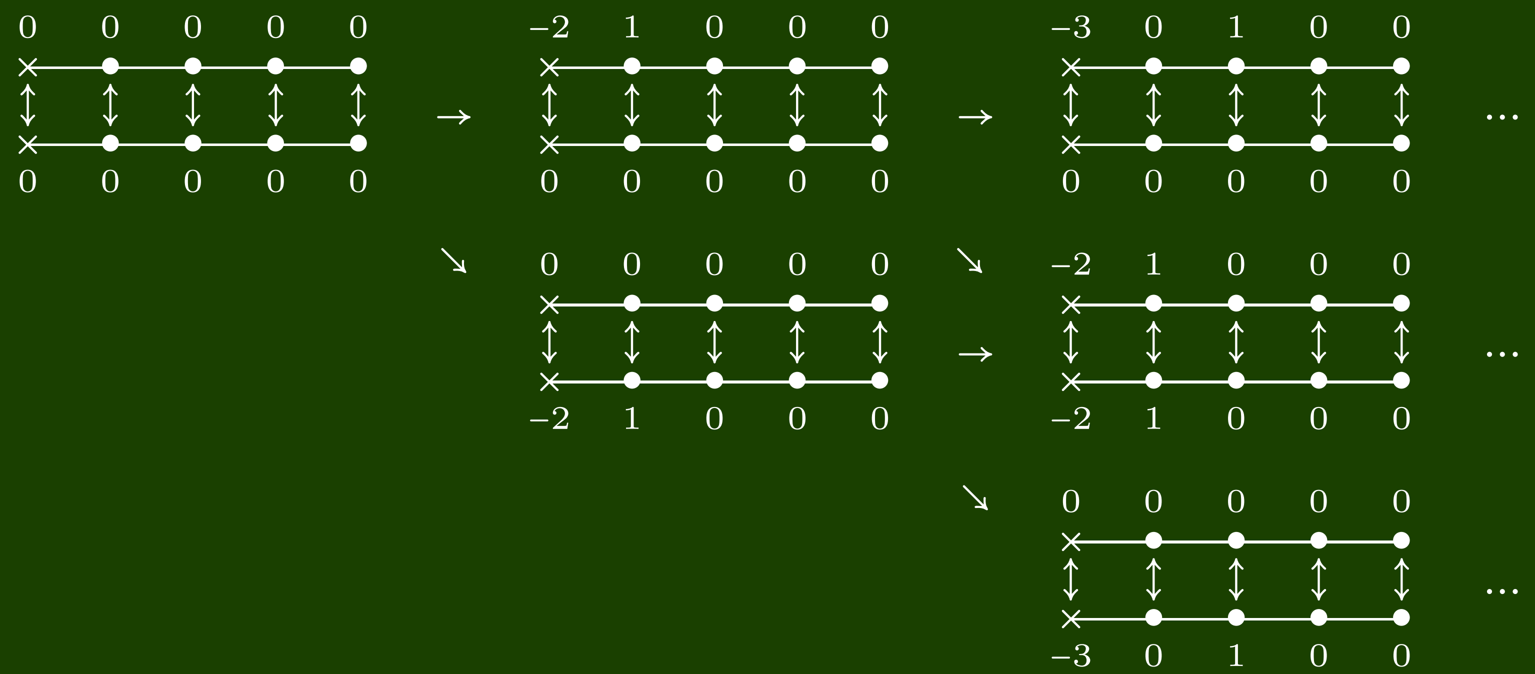


The de Rham complex

Remark in advance: c-projective geometry is $|1|$-graded

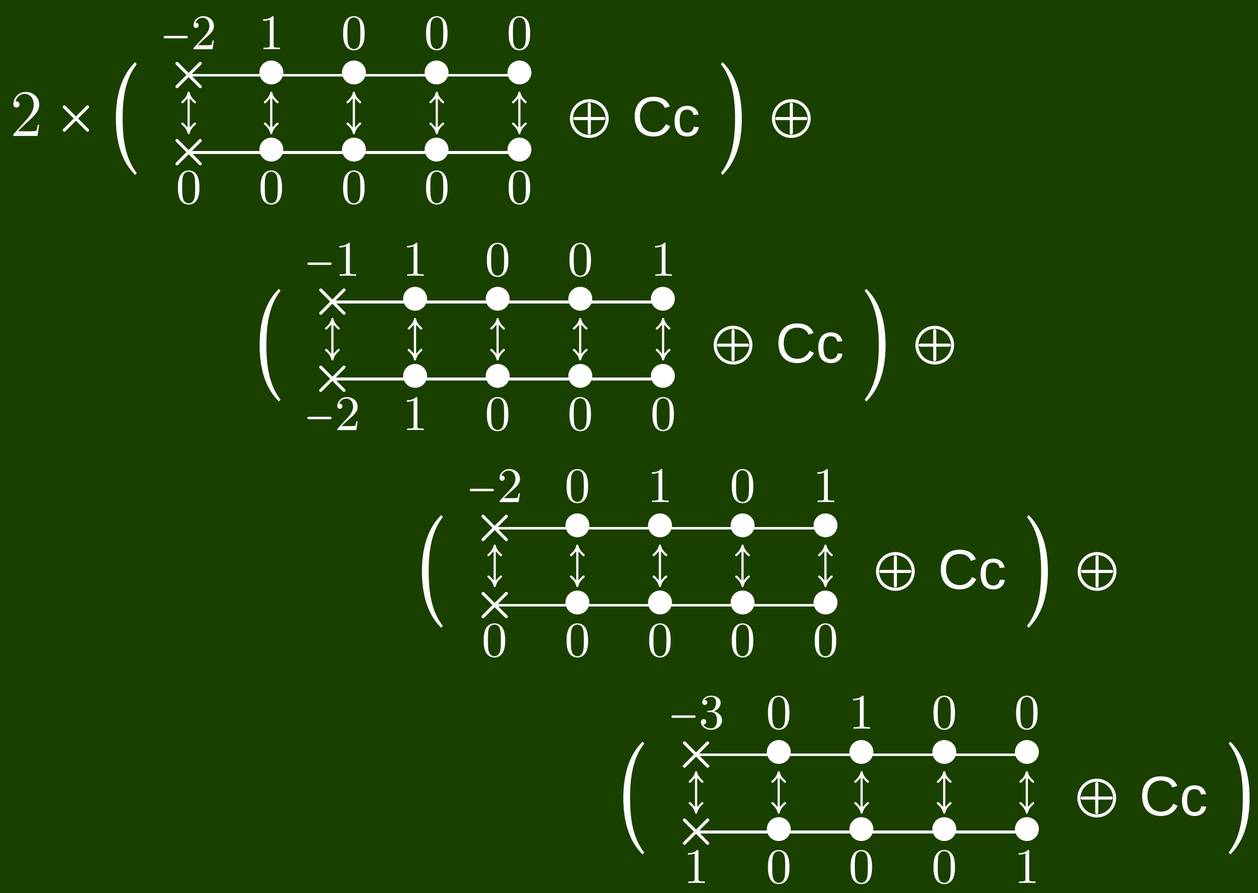
(integrable case)

$$\begin{array}{ccccccc} \Lambda^{0,0} & \to & \Lambda^{1,0} & \to & \Lambda^{2,0} & \cdots \\ & \searrow & & \searrow & & \\ & & \Lambda^{0,1} & \to & \Lambda^{1,1} & \cdots \\ & & & \searrow & & \\ & & & & \Lambda^{0,2} & \cdots \end{array}$$



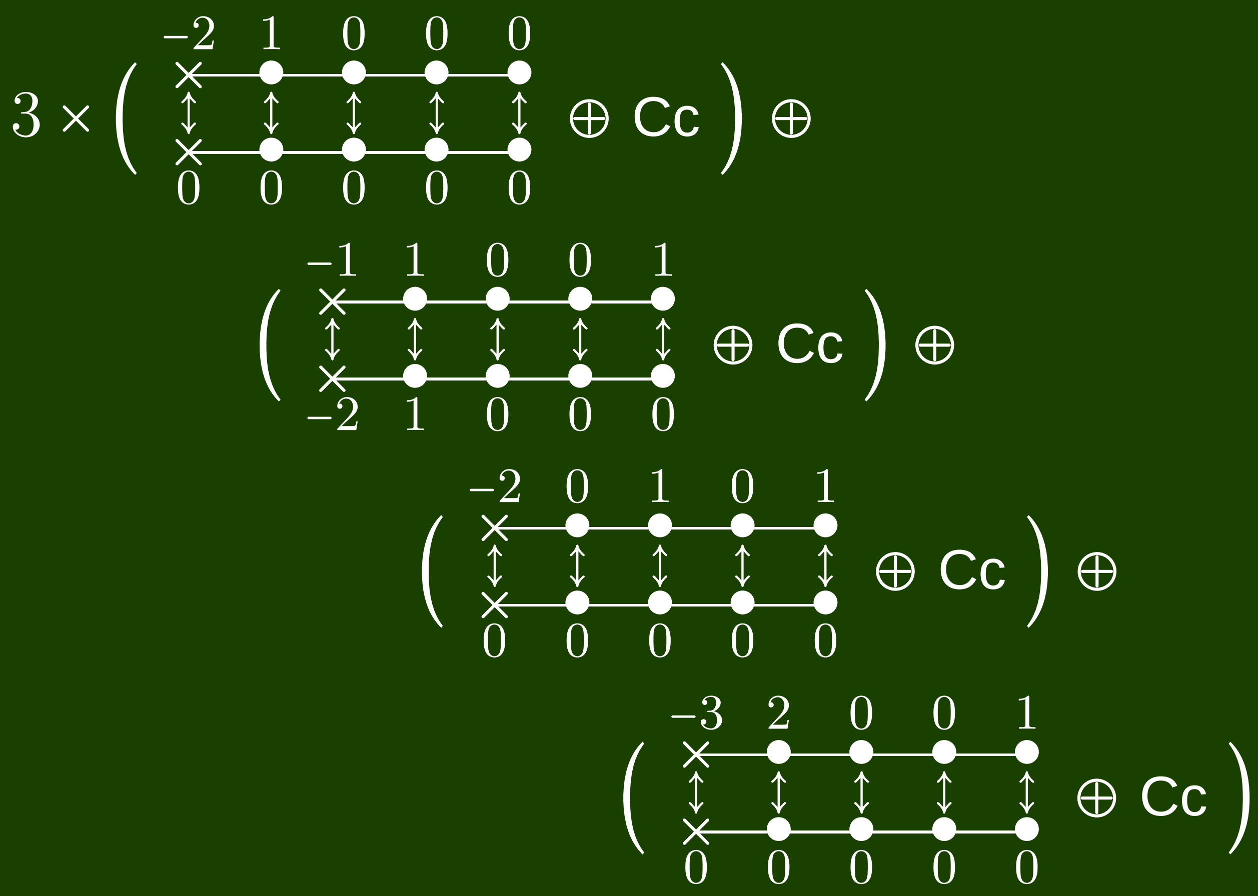
Torsion

$\Lambda^2 \otimes T$ decomposes into five $\mathbb{R}$-irreducibles



Complex connections

$\Lambda^1 \otimes \mathfrak{gl}(n, \mathbb{C})$ decomposes into six $\mathbb{R}$-irreducibles



Compare!

$$0 \to \left(\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathrm{Cc} \right) \oplus \left(\begin{array}{ccccc} -3 & 2 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathrm{Cc} \right) \to$$

$$\Lambda^1 \otimes \mathfrak{gl}(n, \mathbb{C}) \xrightarrow{\partial} \Lambda^2 \otimes TM \to$$

$$\left(\begin{array}{ccccc} -3 & 0 & 1 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{array} \oplus \mathrm{Cc} \right) \to 0,$$

where $\Gamma_{\alpha\beta}{}^{\gamma} \stackrel{\partial}{\longmapsto} 2\Gamma_{[\alpha\beta]}{}^{\gamma}$.

$$\left(\begin{array}{ccccc} -3 & 0 & 1 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{array} \oplus \mathrm{Cc} \right) \ni \text{ the Nijenhuis tensor}$$

C-projective geometry (from thin air)

$$0 \to \left(\begin{smallmatrix} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix} \oplus \text{Cc} \right) \oplus \left(\begin{smallmatrix} -3 & 2 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix} \oplus \text{Cc} \right) \to$$

$$\Lambda^1 \otimes \mathfrak{gl}(n, \mathbb{C}) \xrightarrow{\partial} \Lambda^2 \otimes TM \to$$

$$\left(\begin{smallmatrix} -3 & 0 & 1 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{smallmatrix} \oplus \text{Cc} \right) \to 0,$$

where $\Gamma_{\alpha\beta}{}^{\gamma} \stackrel{\partial}{\longmapsto} 2\Gamma_{[\alpha\beta]}{}^{\gamma}$.

$$\left(\begin{smallmatrix} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix} \oplus \text{Cc} \right) \ni \text{C-projective freedom}$$

C-projective freedom

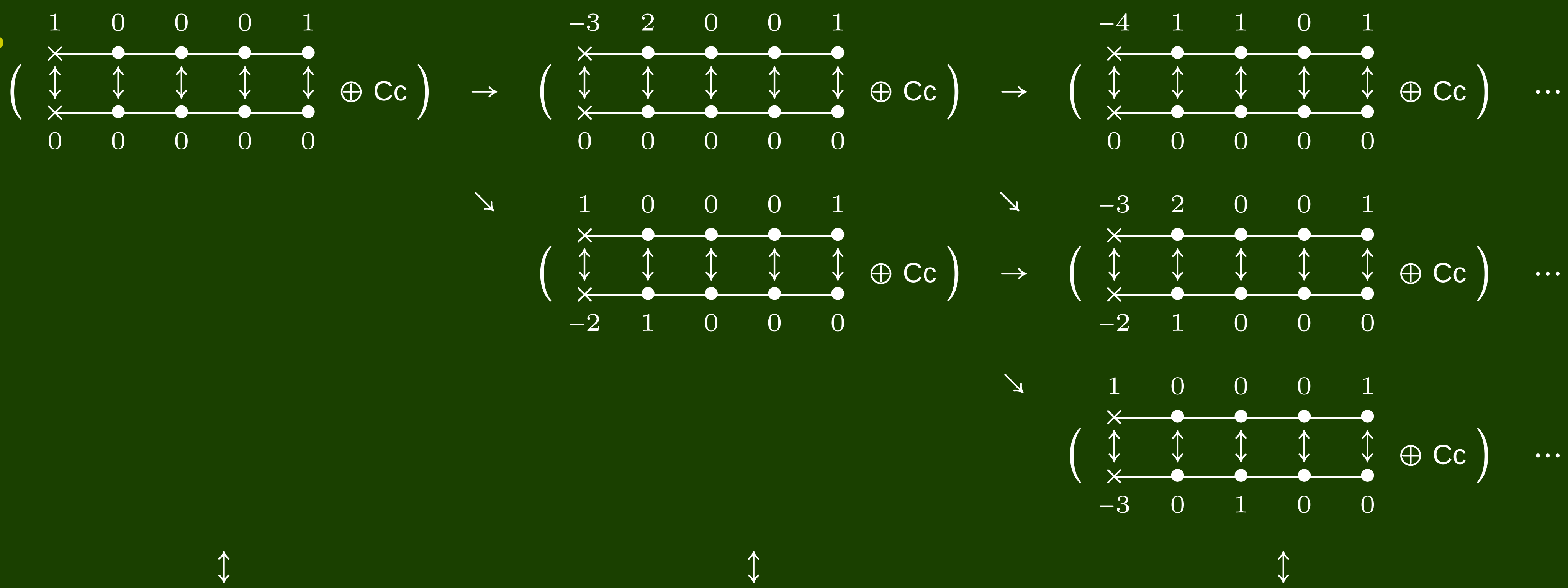
In real money

$$\hat{\nabla}_\alpha \phi_\beta = \nabla_\alpha \phi_\beta - \tfrac{1}{2}(\Upsilon_\alpha \phi_\beta + \Upsilon_\beta \phi_\alpha - J_\alpha{}^\gamma \Upsilon_\gamma J_\beta{}^\delta \phi_\delta - J_\beta{}^\gamma \Upsilon_\gamma J_\alpha{}^\delta \phi_\delta)$$

In complex money (barred and unbarred indices)

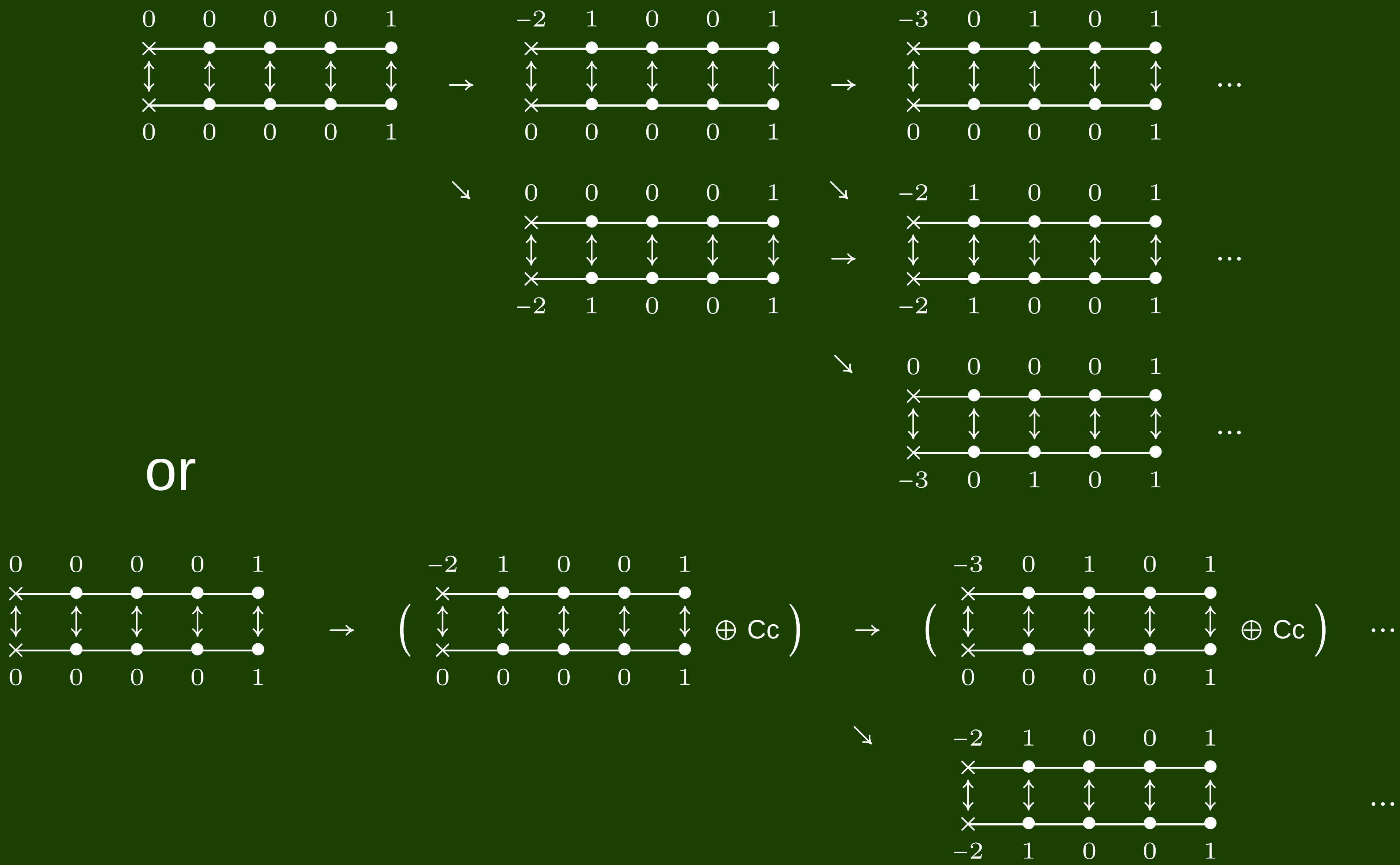
- $\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$
- $\hat{\nabla}_{\bar{a}} \phi_b = \nabla_{\bar{a}} \phi_b$ $\quad (\bar{\partial} : \Lambda^{1,0} \to \Lambda^{0,1} \otimes \Lambda^{1,0})$
- $\hat{\nabla}_{\bar{a}} \phi_{\bar{b}} = \nabla_{\bar{a}} \phi_{\bar{b}} - \Upsilon_{\bar{a}} \phi_{\bar{b}} - \Upsilon_{\bar{b}} \phi_{\bar{a}}$
- $\hat{\nabla}_a \phi_{\bar{b}} = \nabla_a \phi_{\bar{b}}$

Deformation complex



vector fields ⟶ infinitesimal deformations ⟶ harmonic curvature

C-metrisability complex



Parabolic viewpoint

Curvature

Tractors

Prolongation

Kähler metrics

Kähler-Einstein metrics

Hamiltonian two-forms

. . . and more besides . . .

THE END

THANK YOU