The range of the double fibration transform

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[joint work with Joseph Wolf]

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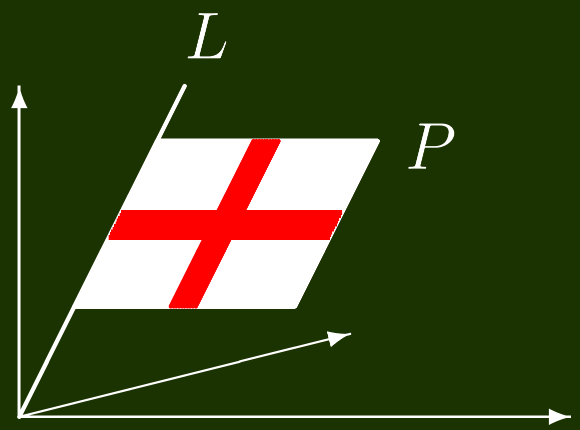
Flag manifolds

- $\mathbb{CP}_n = \{L \subset \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} L = 1\}$
- $\mathrm{Gr}_k(\mathbb{C}^{n+1}) = \{L \subset \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} L = k\}$
- $\mathrm{F}_{1,2}(\mathbb{C}^3) = \{L \subset P \subset \mathbb{C}^3 \mid \dim_{\mathbb{C}} L = 1,\ \dim_{\mathbb{C}} P = 2\}$



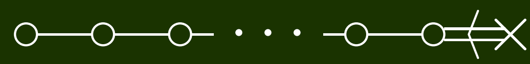
$$\mathbb{CP}_2 = \mathrm{SL}(3,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \qquad \mathrm{F}_{1,2}(\mathbb{C}^3) = \mathrm{SL}(3,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

Flag manifolds cont'd (generalised)

$$\mathrm{SL}(n+1,\mathbb{C})\Big/\left\{\begin{pmatrix} * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix}\right\} =$$

Dynkin diagram with n nodes

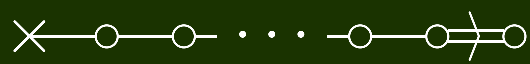
Langrangian Grassmannians

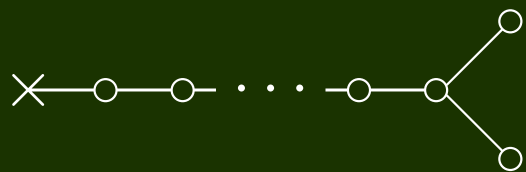


cHss

Quadrics

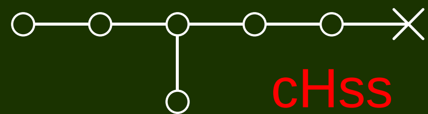
cHss





cHss

Exceptional examples



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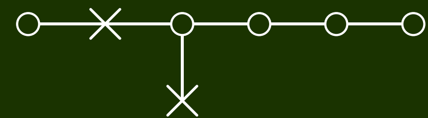


Flag domains

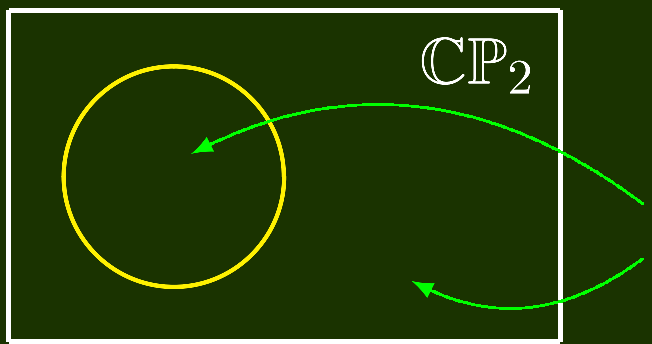
Flag manifold $Z = G/Q$ $\begin{cases} G = \text{complex simple Lie group} \\ Q = \text{complex parabolic subgroup} \end{cases}$

New ingredient $G_0 = \text{real form of } G$. Let it act on Z.

Theorem (Wolf) There are finitely many orbits.

Corollary There are open orbits. ←-- Flag domain $\boxed{D \subseteq Z}$

Example $Z = \mathbb{CP}_2 = \mathrm{SL}(3,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$ $G_0 = \mathrm{SU}(2,1)$



two possible flag domains $D \subset \mathbb{CP}_2$

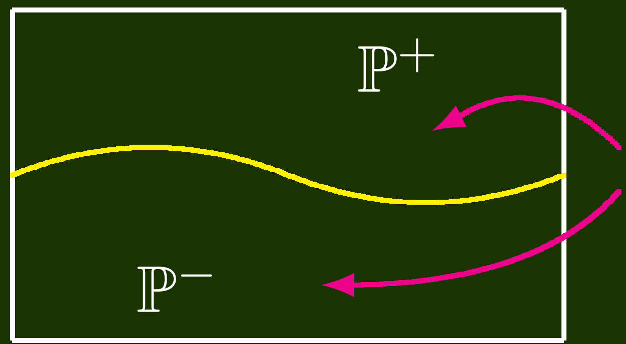
Twistor space

$$\mathbb{CP}_3 = \mathrm{SL}(4,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \right\}$$

$G_0 = \mathrm{SU}(4) = \mathrm{Spin}(6) \Rightarrow D = \mathbb{CP}_3$ one orbit

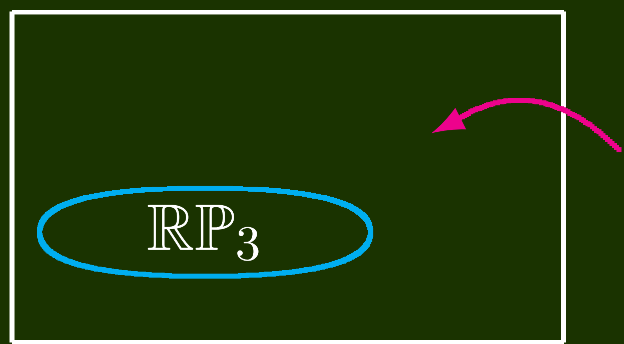
$G_0 = \mathrm{SL}(2,\mathbb{H}) = \mathrm{Spin}(5,1) \Rightarrow D = \mathbb{CP}_3$ one orbit!!

$G_0 = \mathrm{SU}(2,2) = \mathrm{Spin}(4,2)$



two open orbits

$G_0 = \mathrm{SL}(4,\mathbb{R}) = \mathrm{Spin}(3,3)$

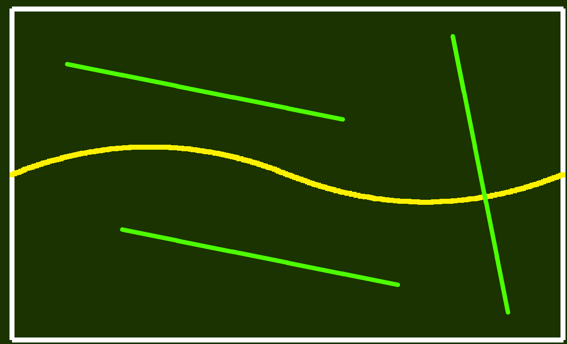


one open orbit

Compactified Minkowski space

$$\mathbb{M} \equiv \mathrm{Gr}_2(\mathbb{C}^4) = \mathrm{SL}(4,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

$$G_0 = \mathrm{SU}(2,2) = \mathrm{Spin}(4,2)$$



three open orbits

Flag domains: $\mathbb{M}^{++}$ $\mathbb{M}^{+-}$ $\mathbb{M}^{--}$
(six orbits in total, three open and one closed)

Closed orbit $=$ compactified Minkowski space
$\mathrm{SU}(2,2)$ acts by conformal automorphisms

$\mathbb{M} =$ complexified compactified Minkowski space

Cycle spaces

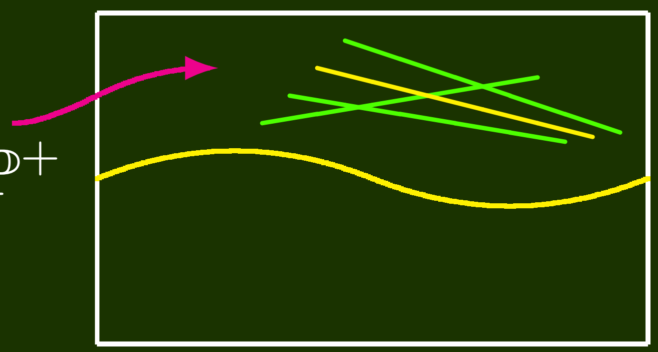
Ingredients so far

- flag manifold $Z = G/Q$ 
- flag domain $D \subseteq Z$ for $G_0 \subset G$

Let $K_0 =$ a maximal compact subgroup of G_0.
EG $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)) \subset \mathrm{SU}(2,2)$

Theorem (Wolf) There is exactly one K_0-orbit $C_0 \subset D$ that is a complex submanifold. ←-- base cycle

Definition $\mathcal{M}_D \equiv \{gC_0 \subset D \mid g \in G\}^\circ$ ←-- cycle space

EG $D = \mathbb{P}^+$  $\Longrightarrow \mathcal{M}_D = \mathbb{M}^{++}$

(maybe misleading)

Double fibrations

Correspondence

$$\mathcal{X}_D = \{(z, C) \in D \times \mathcal{M}_D \mid z \in C\}$$

$$\begin{array}{ccccc} & & \mathcal{X}_D & & \\ & \swarrow^{\mu} & & \searrow^{\nu} & \\ G/Q = Z \supseteq D & & & & \mathcal{M}_D \end{array}$$

$$C_m = \mu(\nu^{-1}(m))$$

Twistor Correspondence

$$\begin{array}{ccccccccccc} & & \mathbb{X}^{++} & & & & & & \mathbb{F}_{1,2}(\mathbb{C}^4) & & \\ & \swarrow^{\mu} & & \searrow^{\nu} & & \subset & & \swarrow^{\mu} & & \searrow^{\nu} & \\ \mathbb{P}^{+} & & & & \mathbb{M}^{++} & & \mathbb{CP}_3 & & & & \mathrm{Gr}_2(\mathbb{C}^4) \end{array}$$

Types of flag domains/cycle spaces

$\mathcal{M}_Z \equiv \{gC_0 \mid g \in G\} = G/J,$ where $J \equiv \{g \in G \mid gC_0 = C_0\}$

NB $J \supseteq K$ (often equal) $\therefore$ $\dim_{\mathbb{C}} \mathcal{M}_Z \leq \dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} K_0$

${\mathcal{M}_D}^{\mathrm{open}} \subset \mathcal{M}_Z \rightsquigarrow \mathcal{M}_D$ is holomorphic

$\mathcal{M}_D$ might not be G_0-homogeneous!!

Perhaps: $D = Z$ (G_0 compact or rare cases).

Otherwise: trichotomy via decomposition of K-modules

$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+$ and $\mathfrak{j} = \mathfrak{k} \oplus \mathfrak{s}_+$ Eg: $\mathrm{SU}(2,2)$ on $\mathbb{P}^+$

$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+$ and $\mathfrak{j} = \mathfrak{k}$ Eg: $\mathrm{SU}(2,2)$ on $\mathbb{M}^{+-}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and $\mathfrak{j} = \mathfrak{k}$ Eg: $\mathrm{SL}(4,\mathbb{R})$ on $\mathbb{CP}_3 \setminus \mathbb{RP}_3$

Hermitian holomorphic case

$\mathfrak{g} = \mathfrak{s}_- \oplus \mathfrak{k} \oplus \mathfrak{s}_+$ and $\mathfrak{j} = \mathfrak{k} \oplus \mathfrak{s}_+$ Then

- $\mathfrak{s}_- = (\mathfrak{s}_+)^*$ Abelian subalgebras of $\mathfrak{g}$
- $\mathfrak{j}$ is parabolic, $J \equiv P$ and $\mathcal{M}_Z = G/P$ is a flag manifold
- $\mathcal{M}_Z$ is a compact Hermitian symmetric space
- $\mathcal{M}_D \subset \mathcal{M}_Z = G/P$ is G_0-homogeneous, i.e. a flag domain
- $\mathcal{M}_D = G_0/K_0$ is a bounded symmetric domain
- $\dim_{\mathbb{C}} \mathcal{M}_D = (\dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} K_0)/2$

$$D \xleftarrow{\mu} \mathcal{X}_D \xrightarrow{\nu} \mathcal{M}_D \quad \subset \quad Z = G/Q \xleftarrow{\mu} G/(Q \cap P) \xrightarrow{\nu} G/P = \mathcal{M}_Z$$

all flag manifolds

Double fibration transform: generalities

$$\begin{array}{ccc} & \mathcal{X}_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & \mathcal{M}_D \end{array}$$

Theorem (. . . , Fels-Huckleberry-Wolf)

- $\mathcal{M}_D$ is Stein (and contractible)
- the fibres of μ are contractible

Let $s = \dim_{\mathbb{C}} C_0 = \dim_{\mathbb{C}}$(fibres of ν).
Let $\mathbb{E} \to Z \supset D$ be a homogeneous vector bundle on $Z = G/Q$.

Then $m \mapsto H^s(C_m, \mathcal{O}(\mathbb{E}|_{C_m}))$ is a hvb on $G/J = \mathcal{M}_Z \supset \mathcal{M}_D$.
Call it $\mathbb{E}'$ $\quad$ ($\mathcal{O}(\mathbb{E}') \equiv \nu_*^s(\mu^*\mathcal{O}(\mathbb{E}))$)

Double fibration transform $\mathcal{P} : H^s(D, \mathcal{O}(\mathbb{E})) \to \Gamma(\mathcal{M}_D, \mathcal{O}(\mathbb{E}'))$

$H^s(D, \mathcal{O}(\mathbb{E})) \ni \omega \mapsto \omega|_{C_m} \in H^s(C_m, \mathcal{O}(\mathbb{E}|_{C_m}))$ for $m \in \mathcal{M}_D$

cf. Radon transform or X-ray transform

Double fibration transform: specifics

Example $\quad G = \mathrm{SL}(4,\mathbb{C}) \quad Z = \mathbb{CP}_3 \quad G_0 = \mathrm{SU}(2,2)$

$D = \mathbb{P}^+ \qquad \mathcal{M}_D = \mathbb{M}^{++} \qquad \mathbb{E} = \Omega^3$ [i.e. $\mathcal{O}(\mathbb{E}) = \mathcal{O}(-4)$]

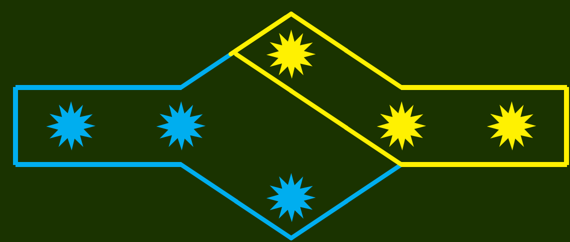
$$\mathbb{E} = \overset{-4}{\times}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ} \implies \mu^*\mathbb{E} = \overset{-4}{\times}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\circ} \implies \mathbb{E}' = \nu^1_*\mu^*\mathbb{E} = \overset{2}{\circ}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{0}{\circ} = \Omega^2_+$$

Cohomological machinery $\rightsquigarrow$

$\mathcal{P} : H^1(\mathbb{P}^+, \Omega^3) \to \Gamma(\mathbb{M}^{++}, \Omega^2_+)$ satisfies

- $\mathcal{P}$ is injective
- $\mathrm{range}(\mathcal{P}) = \{\phi \in \Gamma(\mathbb{M}^{++}, \Omega^2_+)$ s.t. $d\phi = 0\}$ (Maxwell).

$$\Omega^0 \xrightarrow{d} \Omega^1 \begin{matrix} \nearrow & \Omega^2_+ & \searrow \\ \searrow & \Omega^2_- & \nearrow \end{matrix} \Omega^3 \to \Omega^4$$



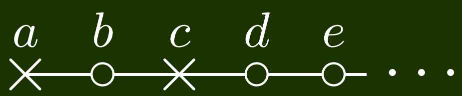
Double fibration transform: injectivity

Theorem (Huckleberry & Wolf) $\mathbb{E}$ 'sufficiently negative'

$$\Rightarrow \quad \boxed{\mathcal{P} : H^s(D, \mathcal{O}(\mathbb{E})) \to \Gamma(\mathcal{M}_D, \mathcal{O}(\mathbb{E}')) \text{ is injective}}$$

Theorem (E & Wolf) In the Hermitian holomorphic case, $\mathbb{E}$ 'sufficiently negative'

$$\Leftrightarrow \quad \boxed{\Omega_Z^{\mathrm{top}} \otimes \mathbb{E}^* \text{ is non-negative}}$$

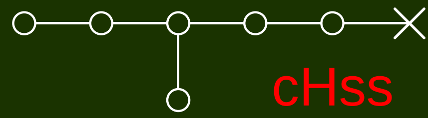
$\Omega_Z^{\mathrm{top}} \otimes \mathbb{E}^* =$ 

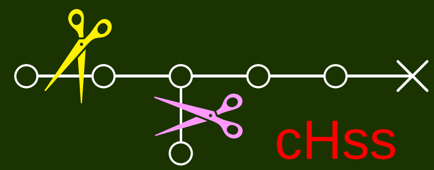
$\boxed{a, c, \ldots \geq 0}$

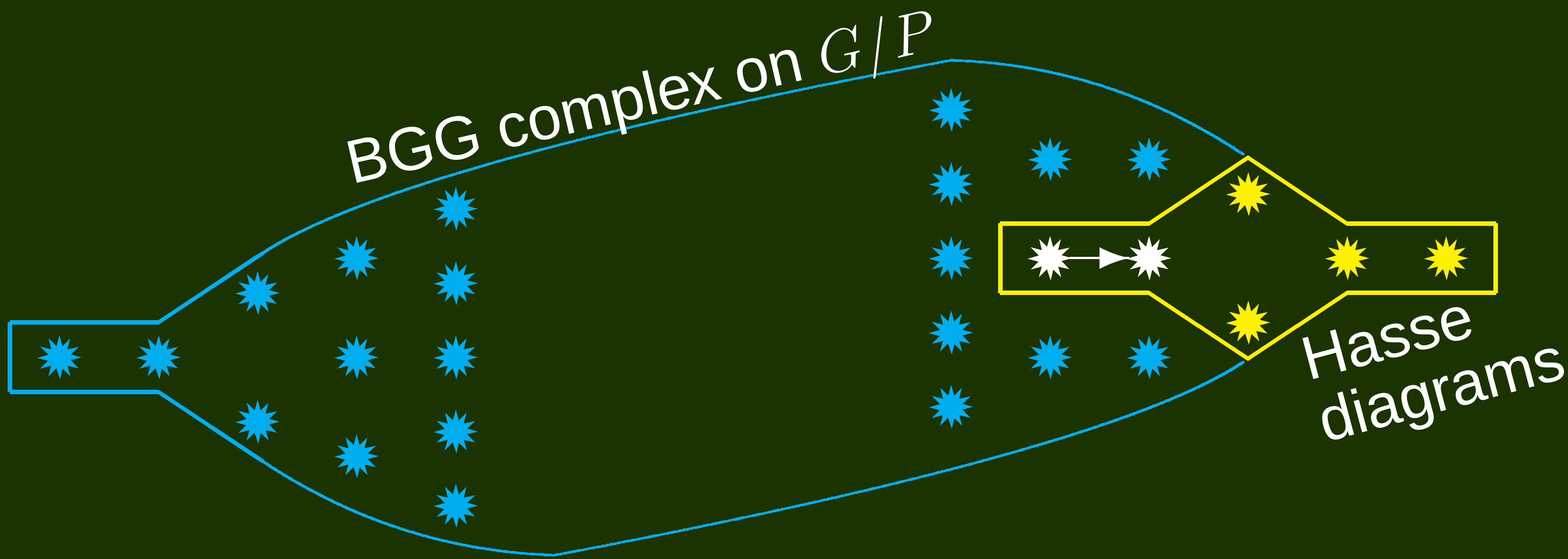
$\mathbb{E} \rightsquigarrow \Omega_Z^{\mathrm{top}} \otimes \mathbb{E}^*$ OK via diagrams (effective algorithm).

Double fibration transform: range

Theorem (E & Wolf) In the Hermitian holomorphic case,

$\mathcal{M}_D \subset \mathcal{M}_Z = G/P =$ 

(Fibres of μ) $\subset Q/(P \cap Q) =$ 



THANK YOU