Conformally Fedosov Manifolds

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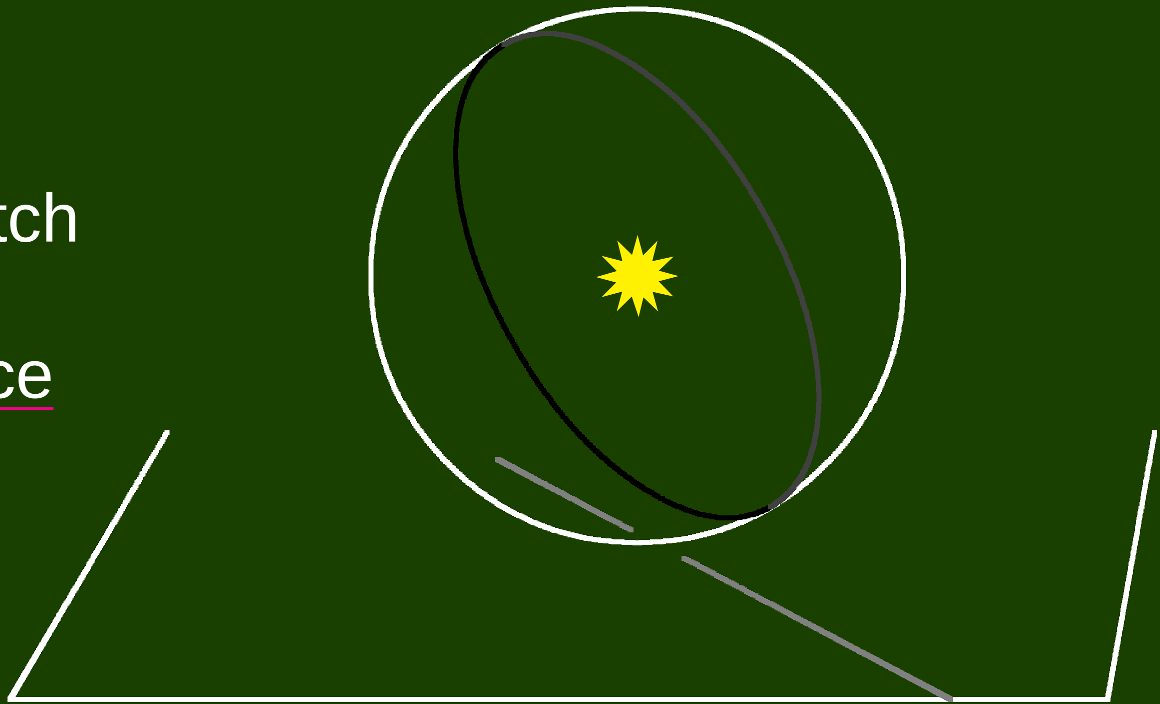
Projective differential geometry

Defn $\hat{\nabla} \sim \nabla \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) The round sphere is projectively flat



Affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a projective equivalence

Operational Defn

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

Conformally symplectic geometry

symplectic geometry: $\left\{ \begin{array}{l} \bullet \text{ non-degenerate } 2\text{-form } J \\ \bullet\ J \text{ is closed: } dJ = 0 \end{array} \right.$

conformally „ „ : $\left\{ \begin{array}{l} \bullet \text{ non-degenerate } 2\text{-form } J \\ \bullet\ J \text{ is conformally closed: } dJ = 2\alpha \wedge J \end{array} \right.$

(where α is a closed 1-form)

$$J \mapsto \hat{J} = \Omega^2 J \quad \rightsquigarrow \quad \alpha \mapsto \hat{\alpha} = \alpha + \Upsilon \qquad \text{where } \Upsilon = d \log \Omega$$

Example

$$J \equiv (1/\|x\|)^2 \, (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \cdots) \quad \text{on } \mathbb{R}^{2n} \setminus \{0\}$$

is invariant under dilation $x \mapsto \lambda x$
so descends to $S^1 \times S^{2n-1}$

Combine!

Recall: $\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$

NB: $\hat{\nabla}_{(a} J_{b)c} = \nabla_{(a} J_{b)c} - 3\Upsilon_{(a} J_{b)c}$

Decree (for conformally Fedosov)

- $[J_{ab}]$ conformally symplectic
- $[\nabla_a]$ projective structure
- $\nabla_{[a} J_{bc]} = 2\alpha_{[a} J_{bc]} \quad (\nabla_{[a}\alpha_{b]} = 0)$
- $\nabla_{(a} J_{b)c} = \beta_{(a} J_{b)c}$

Normalise: $\alpha_a = \beta_b$. Then $\boxed{\nabla_a J_{bc} = 2J_{a[b}\alpha_{c]}}$ quite strong

Remaining conformal change

$$\hat{J}_{ab} = \Omega^2 J_{ab} \qquad \hat{\alpha}_a = \alpha_a + \Upsilon_a \qquad \Upsilon_a = \nabla_a \log \Omega$$
$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

Fedosov gauge

Locally: choose $\alpha_a = 0$ $\quad (\iff \nabla_a J_{bc} = 0)$

Remaining freedom

$$\hat{J}_{ab} = \Omega^2 J_{ab} \text{ where } \Omega \text{ is constant (local)}$$

$$\hat{\nabla}_a = \nabla_a \text{ canonically defined global connection}$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c \equiv R_{ab}{}^c{}_d X^d$$

$R_{abcd} \equiv R_{ab}{}^e{}_d J_{ec} \implies$

$$\boxed{R_{abcd} = R_{[ab](cd)} \qquad R_{[abc]d} = 0}$$

$\implies$

$$R_{abcd} = V_{abcd} + J_{ac}\Phi_{bd} - J_{bc}\Phi_{ad} + J_{ad}\Phi_{bc} - J_{bd}\Phi_{ac} + 2J_{ab}\Phi_{cd}$$

$\uparrow$ J-trace-free $\qquad$ $\uparrow$ symmetric $\qquad$ cf. Weyl + Schouten

Complex projective space

With Fubini-Study metric $\rightsquigarrow$

$$R_{ab}{}^c{}_d = \delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} + J_{ad}J^c{}_b - J_{bd}J^c{}_a + 2J_{ab}J^c{}_d$$

so

$$R_{abcd} = J_{ac}g_{bd} - J_{bc}g_{ad} + J_{ad}g_{bc} - J_{bd}g_{ac} + 2J_{ab}g_{cd}$$

$\Longrightarrow$

$$\boxed{V_{abcd} = 0 \qquad \Phi_{ab} = g_{ab}}$$

Projective viewpoint (Conformal viewpoint = exercise)

$$R_{ab}{}^c{}_d = J_{ad}J^c{}_b - J_{bd}J^c{}_a + 2J_{ab}J^c{}_d - \tfrac{3}{2n-1}(\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad}) + 2\tfrac{n+1}{2n-1}(\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad})$$

Projective Weyl tensor !!

Conformal differential geometry

Densities of weight w $\quad g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab} \implies \hat{\phi} = \Omega^w \phi$

Tractors $\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$ transform according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b - \frac{1}{2} \Upsilon^b \Upsilon_b \sigma \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} \Upsilon_b \equiv \nabla_b \log \Omega \\ \Upsilon^b = g^{bc} \Upsilon_c \end{array}$$

Tractor connection (É. Cartan / T.Y. Thomas)

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + \mathrm{P}_{ab} \sigma \\ \nabla_a \rho - \mathrm{P}_a{}^b \mu_b \end{bmatrix}$$

flat $\iff$ Weyl $= 0$
[Cotton-York $= 0$ when $\dim = 3$]

Structure group $= \mathrm{SO}(n+1,1)$

Conformally Fedosov tractors

Densities of weight w $\quad J_{ab} \mapsto \hat{J}_{ab} = \Omega^2 J_{ab} \implies \hat{\phi} = \Omega^w \phi$

Tractors $\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$ transform according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b + \Upsilon^b \alpha_b \sigma \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} \Upsilon_b \equiv \nabla_b \log \Omega \\ \Upsilon^b \equiv J^{bc} \Upsilon_c \end{array}$$

Tractor connection in Fedosov gauge

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + \Phi_{ab} \sigma \\ \nabla_a \rho - \Phi_a{}^b \mu_b - S_a \sigma \end{bmatrix} \qquad \begin{array}{l} \Phi_a{}^b = J^{bc} \Phi_{ac} \\ S_a = \frac{1}{2n+1} \nabla^b \Phi_{ab} \end{array}$$

Structure group $= \mathrm{Sp}(2n+2, \mathbb{R})$

Tractor curvature

In Fedosov gauge

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix}\sigma\\ \mu_c\\ \rho\end{bmatrix} = \begin{bmatrix}0\\ V_{abcd}\mu^d + Y_{abc}\sigma\\ Y_{abc}\mu^c - \frac{1}{2n}(\nabla^c Y_{abc} - V_{abce}\Phi^{ce})\sigma\end{bmatrix}$$

$$- 2J_{ab}\begin{bmatrix}\rho\\ S_c\sigma - \Phi_{cd}\mu^d\\ S_c\mu^c - \frac{1}{2n}(\Phi_{de}\Phi^{de} + \nabla^c S_c)\sigma\end{bmatrix}$$

where (cf. Cotton-York)

$$Y_{abc} \equiv \nabla_a\Phi_{bc} - \nabla_b\Phi_{ac} + J_{ac}S_b - J_{bc}S_a + 2J_{ab}S_c$$

Bianchi $\Longrightarrow$ $\nabla^d V_{abcd} + (2n+1)Y_{abc} = 0$ so $V_{abcd} = 0 \Longrightarrow$ this part vanishes

Consequences of $V_{abcd} \equiv 0$

$$\boxed{V_{abcd} = 0 \iff (\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma = 2J_{ab}\Theta\Sigma}$$

Bianchi $\implies \nabla_{[a}(J_{bc]}\Theta) = 0$

$\implies$ in Fedosov gauge $J_{[bc}\nabla_{a]}\Theta = 0 \implies \boxed{\nabla_a \Theta = 0}$

$$\Theta \in \Gamma(\mathrm{End}(\mathbb{T})) \cong \Gamma(\otimes^2 \mathbb{T}) \rightsquigarrow \Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Phi_{bc} & S_b \\ 0 & S_c & -\frac{1}{2n}(\Phi^{de}\Phi_{de} + \nabla^c S_c) \end{bmatrix}$$

$$\nabla_a \Theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\nabla_a \Phi_{bc} - J_{ab}S_c - J_{ac}S_b & * \\ 0 & * & * \end{bmatrix}$$

Therefore $\boxed{V_{abcd} = 0 \implies \nabla_a \Phi_{bc} + J_{ab}S_c + J_{ac}S_b = 0}$

Further consequences of $V_{abcd} \equiv 0$

In Fedosov gauge

$$V_{abcd} = 0 \Rightarrow \nabla_a \Theta = 0 \quad \Longleftrightarrow \quad \nabla_a \Phi_{bc} + J_{ab} S_c + J_{ac} S_b = 0$$

$$\Longleftrightarrow \quad \nabla_a \Phi^{bc} + \delta_a{}^b S^c + \delta_b{}^c S^b = 0$$

$$\Longleftrightarrow \quad \underbrace{\boxed{\text{trace-free-part}(\nabla_a \Phi^{bc} = 0)}}_{\text{mobility equations}}$$

$$\Longleftrightarrow \quad \underbrace{\boxed{\begin{aligned} \nabla_a \Phi^{bc} + \delta_a{}^b S^c + \delta_b{}^c S^b &= 0 \\ \nabla_a S^b + \delta_a{}^b X - \Phi_{ac} \Phi^{bc} &= 0 \\ \nabla_a X - 2\Phi_{ab} S^b &= 0 \end{aligned}}}_{\text{extremely strong!!}}$$

Examples with $V_{abcd} \equiv 0$

- $\mathbb{CP}_n$ recall $\boxed{V_{abcd} = 0}$ and $\boxed{\Phi_{ab} = g_{ab}}$ $\rightsquigarrow$

$$\Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Phi_{bc} & S_b \\ 0 & S_c & -\frac{1}{2n}(\Phi^{de}\Phi_{de} + \nabla^c S_c) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -g_{bc} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- $S^1 \times S^{2n-1}$ with $J_{ab} = (1/\|x\|)^2(dx^1 \wedge dx^2 + \cdots)$
  $\nabla_a =$ the flat connection $\rightsquigarrow$

locally $\boxed{\begin{array}{c} V_{abcd} = 0 \\ \Phi_{ab} = 0 \end{array}}$ $\Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Any more?

(Potential) applications

- Integral geometry on $\mathbb{CP}_n$ (with H. Goldschmidt)

Elliptic complex on any (conformally) symplectic manifold

$$\begin{array}{ccccccccccccc}
0 & \to & & \xrightarrow{d} & & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d_\perp} & \cdots & \xrightarrow{d_\perp} & \Lambda^{n-1}_\perp & \xrightarrow{d_\perp} & \Lambda^n_\perp \\
 & & & & & & & & & & & & \downarrow d^{(2)}_\perp \\
0 & \leftarrow & \Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda^2_\perp & \xleftarrow{d_\perp} & \cdots & \xleftarrow{d_\perp} & \Lambda^{n-1}_\perp & \xleftarrow{d_\perp} & \Lambda^n_\perp
\end{array}$$

$V_{abcd} = 0 \implies$
OK to couple with $\mathbb{T}$

Smith (1976, $n = 2$)
Rumin-Seshadri
Tseng-Yau

- Characterisation of $\mathbb{CP}_n$ (with L. Stolovitch)

Blaschke conjecture . . .

THE END

THANK YOU