Conformal foliations

Michael Eastwood
[joint work with Paul Baird]

Australian National University

Disclaimers and references

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Conjugate functions

$$f = f(q,r,s) \quad g = g(q,r,s) \quad \text{s.t.} \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\|^2 = \|\nabla g\|^2 \end{cases}$$

- $$f = q \quad g = r$$
- $$f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2 + s^2}$$
- $$f = r\frac{q^2 + r^2 + s^2}{r^2 + s^2} \quad g = s\frac{q^2 + r^2 + s^2}{r^2 + s^2}$$
- $$f = \frac{(1 - q^2 - r^2 - s^2)r + 2qs}{r^2 + s^2}$$ $$g = \frac{(1 - q^2 - r^2 - s^2)s - 2qr}{r^2 + s^2}$$

$$\begin{array}{ccc} \mathbb{R}^3 & \hookrightarrow & S^3 \\ & & \downarrow \\ \mathbb{R}^2 & \hookrightarrow & S^2 \end{array}$$

Hopf

The implicit function theorem

If $M \hookrightarrow N$ is a hypersurface ($\leftarrow$ non-singular)
Then $M = \{f = 0\}$ locally ($\leftarrow$ non-degenerate)

OK if M and N are smooth ($\Rightarrow$ f smooth)

OK if M and N are complex ($\Rightarrow$ f holomorphic)

What if M and N are CR? ($\Rightarrow$ f CR)?

Yes if M and N are also real-analytic

No in general!

CR geometry

$H \subseteq TM$ and $J : H \to H$ such that

- $J^2 = -\mathrm{Id}$ $\quad$ ($\Rightarrow \mathrm{rank}_{\mathbb{R}} H$ is even)
- $[H^{0,1}, H^{0,1}] \subseteq H^{0,1}$ where $H^{0,1} \equiv \{X \mid JX = -iX\}$

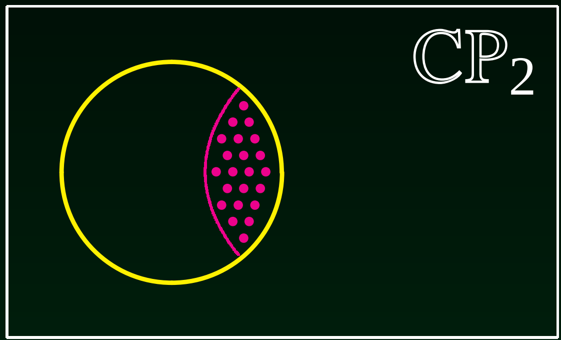
Examples

- $H = TM$ where M is a complex manifold
- $M^{2n-1} \hookrightarrow \mathbb{C}^n$ and $H \equiv TM \cap JTM$, a CR manifold of hypersurface type
- $Q \equiv \{[Z] \in \mathbb{CP}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\}$, the Levi-indefinite hyperquadric in $\mathbb{CP}_3$

$f : M \to \mathbb{C}$ is a CR function $\Leftrightarrow Xf = 0 \; \forall X \in \Gamma(H^{0,1})$

CR functions

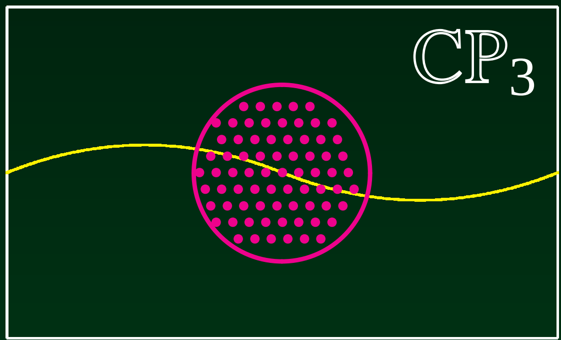
$\{[Z] \in \mathbb{CP}_2 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2\} =$ three-sphere



Theorem (H. Lewy 1956)

CR $\Rightarrow$ holomorphic extension

$\{[Z] \in \mathbb{CP}_3 \mid |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} = Q$



Corollary

CR $\Rightarrow$ holomorphic extension

Hence, a CR function on Q is real-analytic!

CR submanifolds of Q

Suppose $Q \supseteq {}^{\text{open}}\Omega \xrightarrow{f} \mathbb{C}$ is a CR function.

Then $M \equiv \{f = 0\} \subset \Omega$ is a CR submanifold,
i.e. $TM \cap H$ is preserved by J.

Conversely, suppose

- $M \subset \Omega^{\text{open}} \subseteq Q$ is a 3-dim$^{\ell}$ CR submanifold
- M is real-analytic*.

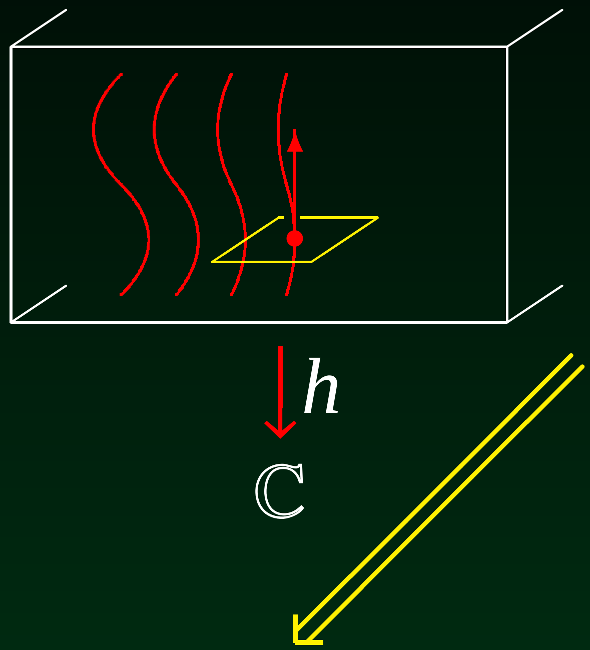
Then

- $M = \tilde{M} \cap Q$ for $\tilde{M} \subset \mathbb{CP}_3$ a complex hypersurface
- $M = \{f = 0\}$ for $f : \Omega \to \mathbb{C}$ CR and real-analytic.

*Q: Can we drop real-analyticity? A: NO

Conformal foliations

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.



U is transversally conformal
$\Leftrightarrow \mathcal{L}_U$ preserves the conformal metric orthogonal to its leaves

isothermal coördinates

$h = f + ig \qquad \langle df, dg \rangle = 0$

$\|df\|^2 = \|dg\|^2$

$\langle dh, dh \rangle = 0$

conjugate functions!

$U \lrcorner\, \omega = 0$

$\langle \omega, \omega \rangle = 0$

$\omega \wedge d\omega = 0$

$dh = \omega \qquad \langle \omega, \omega \rangle = 0$

$d\omega = 0$

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Integrable Hermitian structures

Suppose $J : T\mathbb{R}^4 \to T\mathbb{R}^4$ satisfies

- $J^2 = -\mathrm{Id}$
- $J \in \mathrm{SO}(4)$

$$\Leftrightarrow J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & o & -u \\ w & -v & u & 0 \end{bmatrix}$$

$$u^2 + v^2 + w^2 = 1$$

- $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$ where $T^{0,1} \equiv \{X \mid JX = -iX\}$

$$\mathbb{R}^3 = \{(p, q, r, s) \in \mathbb{R}^4 \mid p = 0\} \subset \mathbb{R}^4$$

Then $U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3} = \left(u\frac{\partial}{\partial q} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial s}\right)\Big|_{\mathbb{R}^3}$

is <u>transversally conformal</u>.

Twistor fibration

$$\begin{array}{ccc} \mathbb{CP}_3 \setminus \{z_3 = z_4 = 0\} & \ni & [z_1, z_2, z_3, z_4] \\ \downarrow \tau & & \mapsto \\ \mathbb{R}^4 & \ni & \begin{bmatrix} p + iq \\ r + is \end{bmatrix} = \dfrac{1}{|z_3|^2 + |z_4|^2} \begin{bmatrix} z_2\bar{z}_3 + z_4\bar{z}_1 \\ z_1\bar{z}_3 - z_4\bar{z}_2 \end{bmatrix} \end{array}$$

$$\tau^{-1}(x) \cong \left\{ J : T_x\mathbb{R}^4 \to T_x\mathbb{R}^4 \;\middle|\; \begin{array}{l} J^2 = -\mathrm{Id} \\ J \in SO(T_x\mathbb{R}^4) \end{array} \right\}$$

Theorem A section $\mathbb{R}^4 \supseteq {}^{\text{open}}\Omega \xrightarrow{J} \mathbb{CP}_3$ of τ defines an integrable Hermitian structure if and only if $\tilde{M} \equiv J(\Omega)$ is a complex submanifold.

Twistor fibration cont'd

$$\begin{array}{ccc} \mathbb{CP}_3 \setminus \{z_3 = z_4 = 0\} & \ni & [z_1, z_2, z_3, z_4] \\ \downarrow \tau & & \downarrow \\ \mathbb{R}^4 & \ni & \begin{bmatrix} p + iq \\ r + is \end{bmatrix} = \dfrac{1}{|z_3|^2 + |z_4|^2} \begin{bmatrix} z_2\overline{z_3} + z_4\overline{z_1} \\ z_1\overline{z_3} - z_4\overline{z_2} \end{bmatrix} \\ \cup & & \\ \mathbb{R}^3 & = \{p = 0\} & \end{array}$$

$$\tau^{-1}(\mathbb{R}^3) = \{[z] \in \mathbb{CP}_3 \mid \Re(z_2\overline{z_3} + z_4\overline{z_1}) = 0\} = Q \setminus \mathbb{I}$$

$$\tau^{-1}(x) \cong \{U \in T_x\mathbb{R}^3 \mid \|U\|^2 = 1\}$$

$Q \setminus \mathbb{I} \cong$ unit sphere bundle

Theorem A section $\mathbb{R}^3 \supseteq^{\text{open}} \Omega \xrightarrow{U} Q$ of $\tau : Q \setminus \mathbb{I} \to \mathbb{R}^3$ defines a conformal foliation if and only if $M \equiv U(\Omega)$ is a CR submanifold.

The story so far

$$
\text{Compactify:}\quad
\begin{array}{ccc}
\mathbb{CP}_3 & \supset & Q = S(TS^3) \\
\downarrow & & \downarrow \\
S^4 & \supset & S^3
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & \mathbb{CP}_3 & \supset & Q & & \\
 & & \cup & & \cup & & \\
\text{complex surface} & = & \tilde{M} & \overset{?}{\supset} & M & = & \text{CR three-fold} \\
 & & \updownarrow & & \updownarrow & & \\
 & & \text{Hermitian structure} & ? & \text{conformal foliation} & &
\end{array}
$$

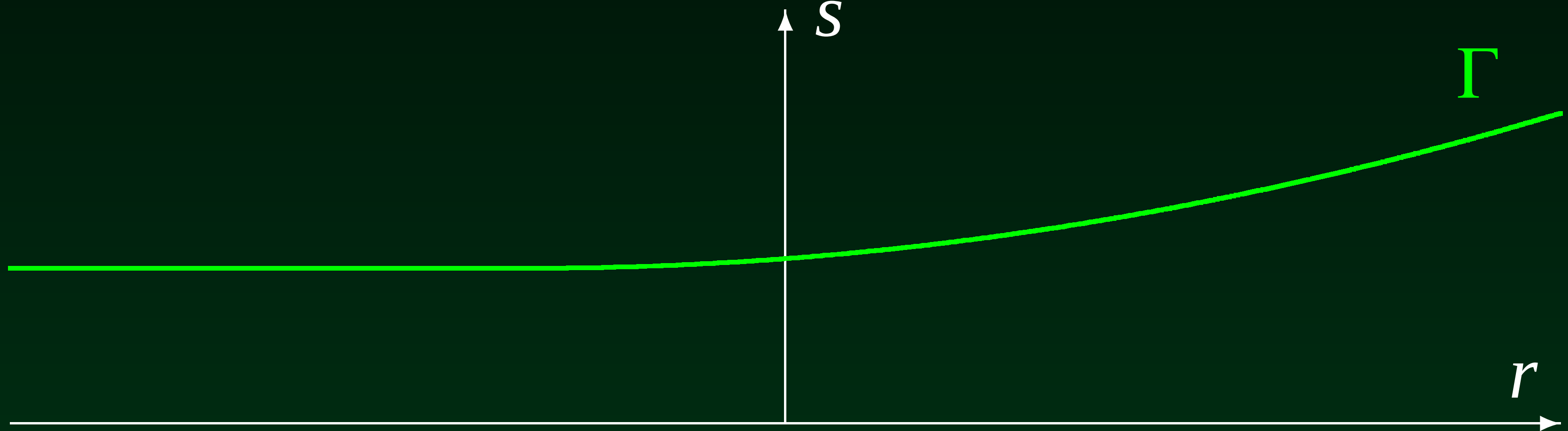
(Proofs: check in local coördinates)

Real-analytic case: $M = \tilde{M} \cap Q$ et cetera

Smooth counterexample

Eikonal equation: $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial s}\right)^2 = 1$

Plenty of non-analytic solutions:



$f =$ signed distance to Γ

$$\left.\begin{array}{lcl} f(q,r,s) & = & f(r,s) \\ g(q,r,s) & = & q \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \langle df, dg\rangle = 0 \\ \|df\|^2 = \|dg\|^2 \end{array}}$$

QED

Real-analytic refinements

$\omega =$ $\mathbb{C}$-valued real-analytic null 1-form on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$

- $\bullet\bullet\bullet$ $\omega \wedge d\omega = 0$
- $\circ\circ\circ$ $\sigma \wedge d\omega = 0 \;\; \forall\, \sigma$ s.t. $\langle \sigma, \omega \rangle = 0$
- $***$ $d\omega = 0$

$*** \Rightarrow \circ\circ\circ \Rightarrow \bullet\bullet\bullet$

$\bullet\bullet\bullet \leftrightarrow \tilde{M} \hookrightarrow \mathbb{CP}_3$

$\circ\circ\circ \leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ (and $\pi(\tilde{S}) = \tilde{M}$)

$*** \leftrightarrow \tilde{S} \hookrightarrow \mathbb{C}^4 \setminus \{0\}$ such that $\tilde{S}$ is Lagrangian

Explicit formulæ

$$\left.\begin{array}{ccccc} Q & \supset & \mathbb{C}\times\mathbb{R}^3 & \ni & (z,q,r,s) \\ \cap & & \cap & & \\ \mathbb{CP}_3 & \supset & \mathbb{C}^3 & \ni & (z,z_1,z_2) \end{array}\right\} \begin{array}{l} z_1 = (r+is)z - iq \\ z_2 = iqz - (r-is) \end{array}$$

$$\boxed{\omega \equiv 2z\,dq + i(1+z^2)\,dr + (1-z^2)\,ds}$$

- $\langle \omega, \omega \rangle = 0$
- $\omega \wedge d\omega = 2\,dz \wedge\ dz_1 \wedge\ dz_2$

$z = z(q,r,s)$ implicitly by $z = \Phi(z_1, z_2)$ holomorphic

$$dz = \frac{\partial\Phi}{\partial z_1}\,dz_1 + \frac{\partial\Phi}{\partial z_2}\,dz_2 \quad \Rightarrow \quad \underbrace{\omega \wedge d\omega = 0}_{\bullet\bullet\bullet}$$

Explicit formulæ cont'd

$$\left.\begin{array}{ccc} \mathbb{C}^2\times\mathbb{R}^3 & \ni & (w,z,q,r,s) \\ \cap & & \\ \mathbb{C}^4 & \ni & (w,z,z_1,z_2) \end{array}\right\} \begin{array}{l} z_1 = (r+is)z - iqw \\ z_2 = iqz - (r-is)w \end{array}$$

$$\boxed{\omega \equiv 2wz\,dq + i(w^2+z^2)\,dr + (w^2-z^2)\,ds}$$

- $\langle\omega,\omega\rangle = 0$
- $$\begin{aligned} d\omega &= 2i(dz\wedge dz_1 - dw\wedge dz_2) \\ &= 2id(z\,dz_1 - w\,dz_2) \end{aligned}$$

$z = z(q,r,s)$ $w = w(q,r,s)$	by	$z\,dz_1 - w\,dz_2 = d(\Xi(z_1,z_2))$ NB: Lagrangian w.r.t. $dz\wedge dz_1 - dw\wedge dz_2$

$\Rightarrow$ $d\omega = 0$ ***

THANK YOU