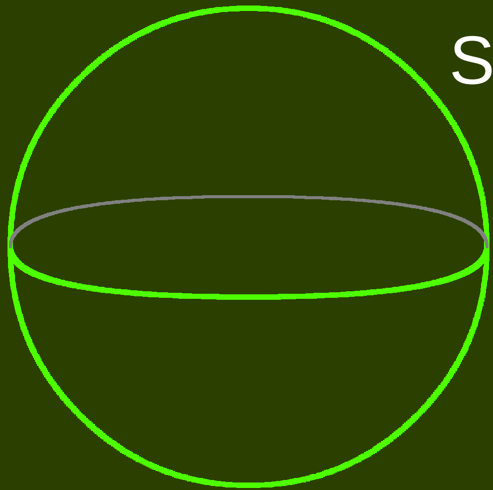
Symmetry and Differential Equations

Michael Eastwood

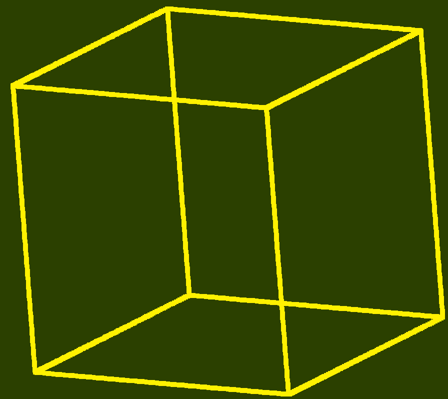
University of Adelaide

Symmetry



Sphere

Cube



Continuous symmetry

Finite symmetry

EG: $G = \mathrm{O}(3)$ or $\mathrm{SO}(3)$

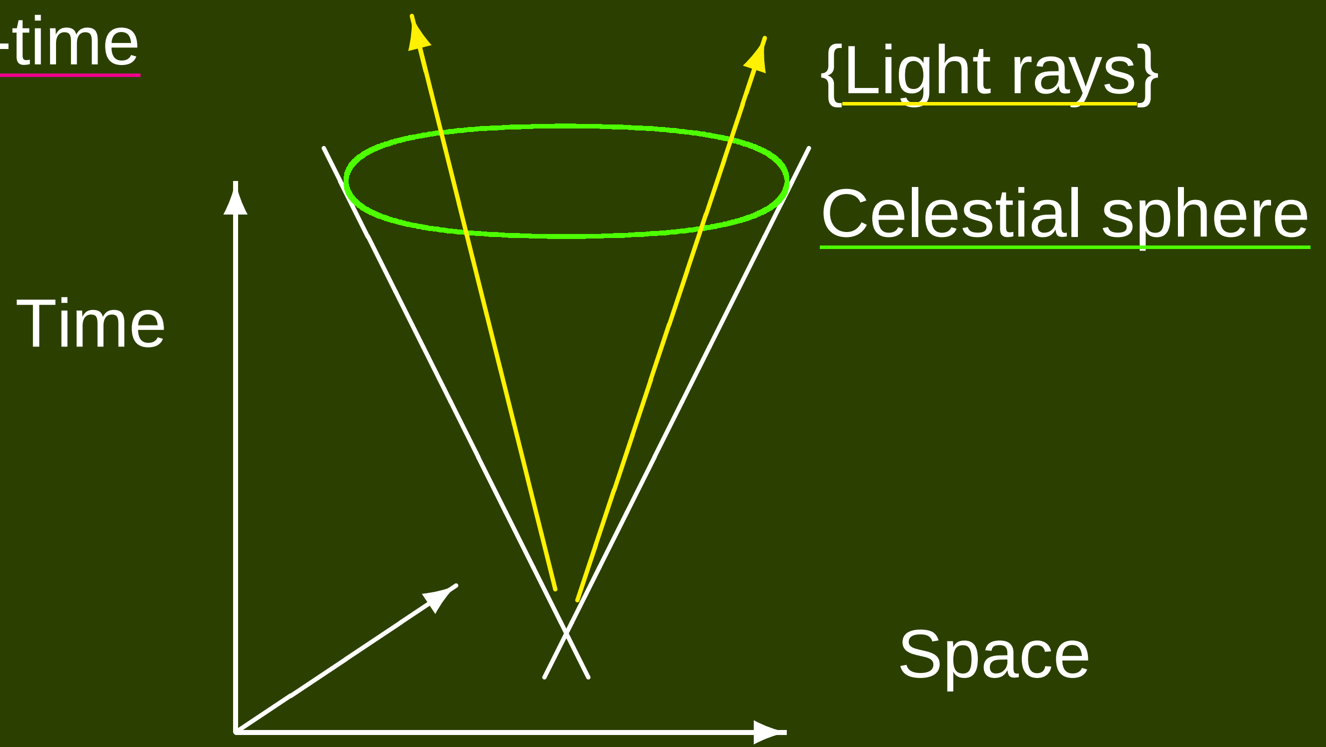
$G = S_4 \times C_2$ or S_4

"spherical symmetry," "spherical polar coördinates," . . .

Lie groups

Symmetry cont'd

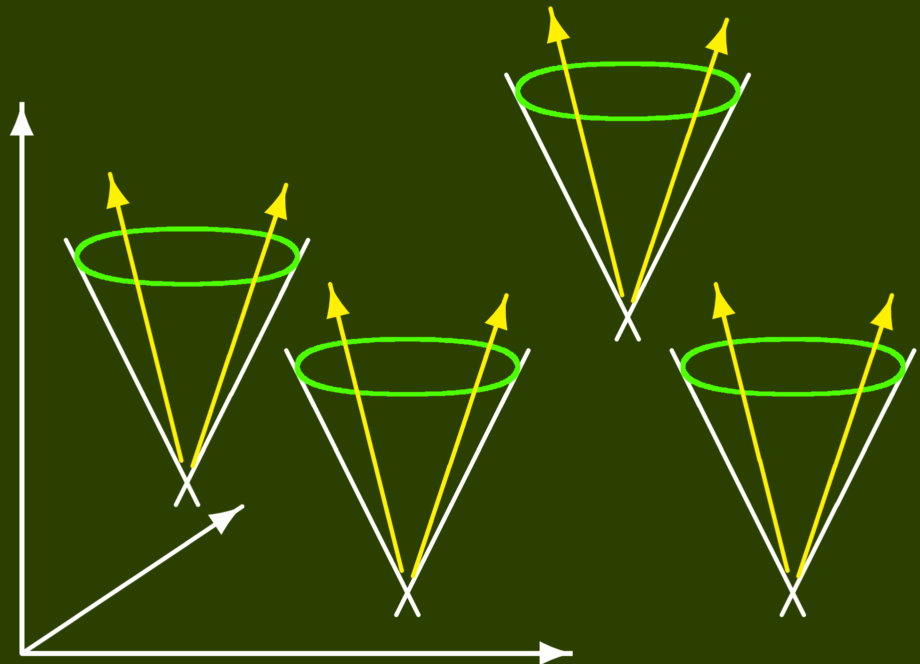
Space-time



Lorentz group $\mathrm{SO}(3,1)$

also acting on the sphere!

Differential Equations



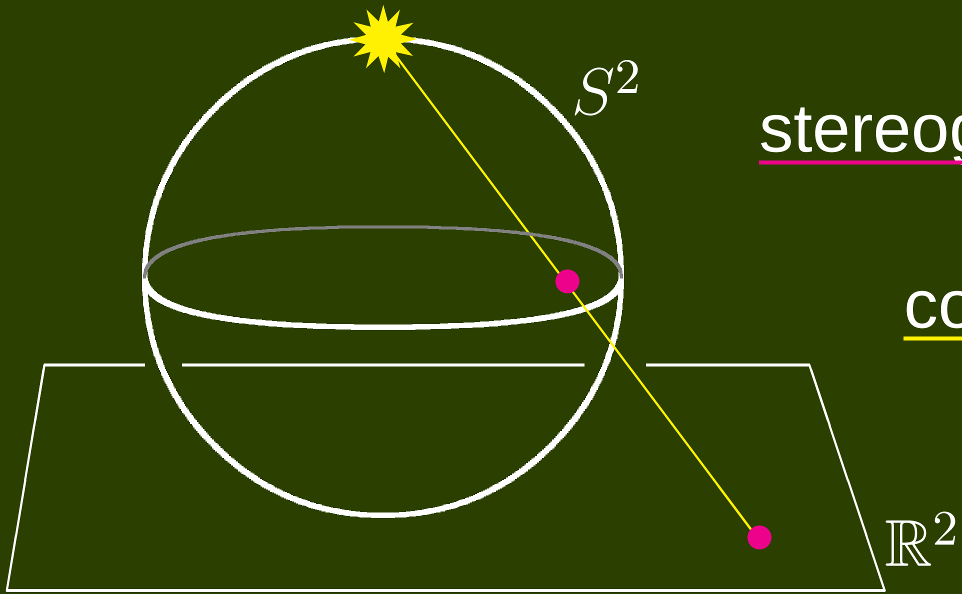
Minkowski, Einstein, . . .

Maxwell "Let there be electromagnetic radiation"

Electric field E
Magnetic field B $\Big\} \rightsquigarrow 2$-form F (NB: $3 + 3 = 6 = \dim \Lambda^2 \mathbb{R}^4$)

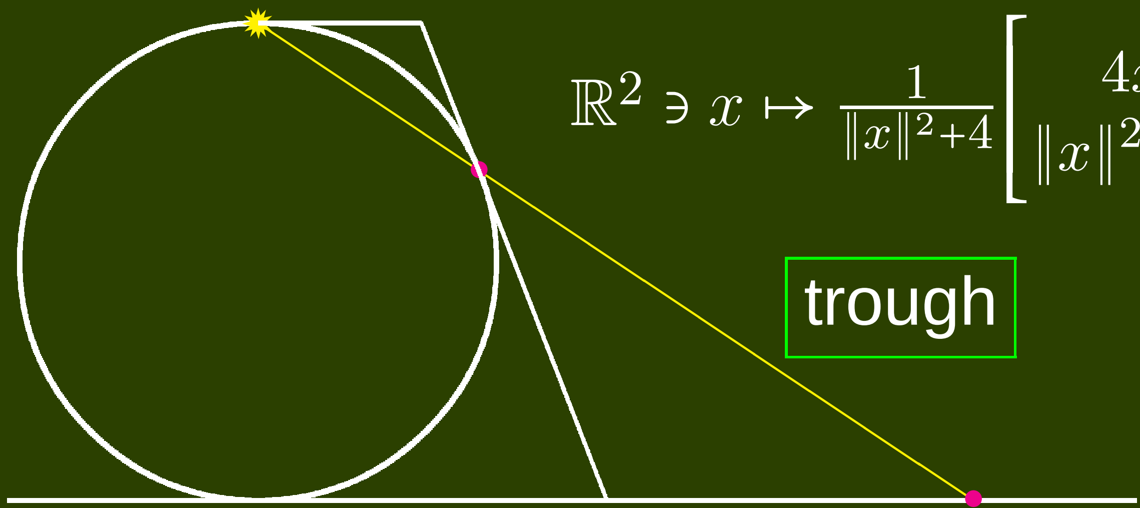
Maxwell's equations $\quad dF = 0 \quad d{*}F = 0$

Two-dimensional conformal geometry



stereographic projection

conformal

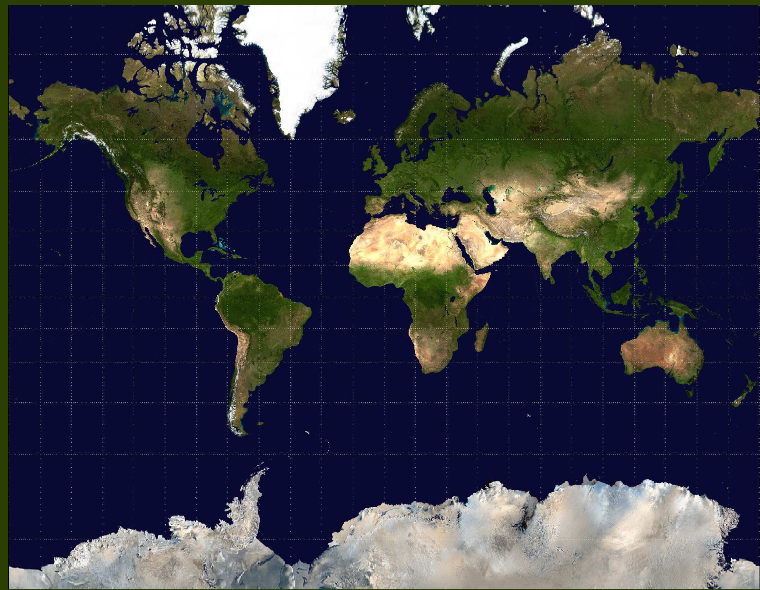


$$\mathbb{R}^2 \ni x \mapsto \frac{1}{\|x\|^2+4}\begin{bmatrix} 4x \\ \|x\|^2 - 4 \end{bmatrix} \in S^2$$

trough

Motivation from navigation

- Mercator (Cartographer) 1569



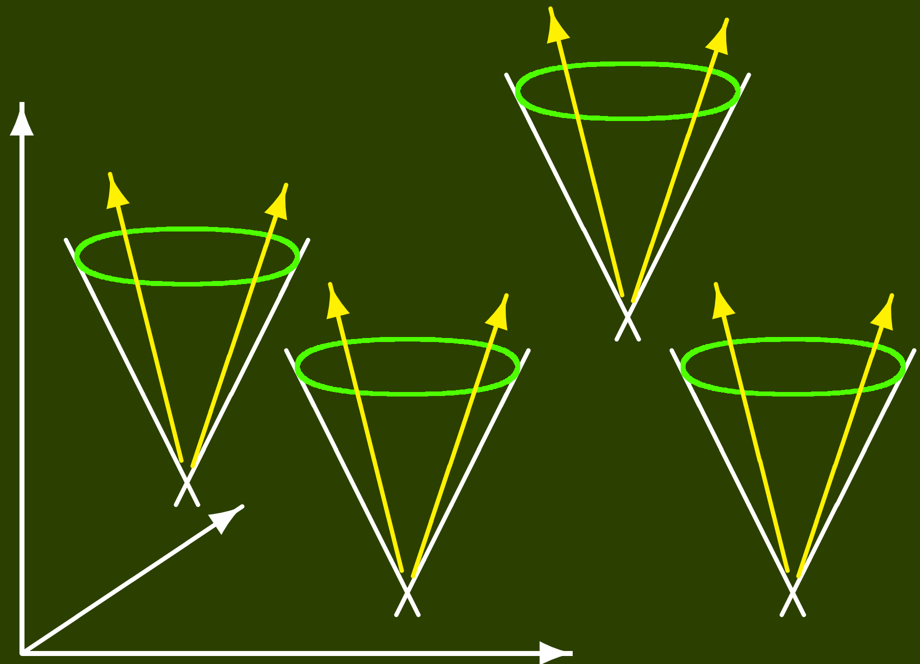
- Wright (Mathematician) 1599

$$S^2 \setminus \{\text{poles}\} \xrightarrow{\text{stereographic}} \mathbb{R}^2 \setminus \{0\} = \mathbb{C} \setminus \{0\} \xrightarrow{\log} \mathbb{C}$$

$$\text{Jac} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \Longleftrightarrow \begin{array}{rcl} u_x & = & v_y \\ v_x & = & -u_y \end{array}$$

Cauchy-Riemann

Four-dimensional conformal geometry



$$\mathrm{SO}(3,1) \cong \mathrm{SL}(2,\mathbb{C})$$

$$\begin{aligned} dF &= 0 \\ d{*}F &= 0 \end{aligned}$$

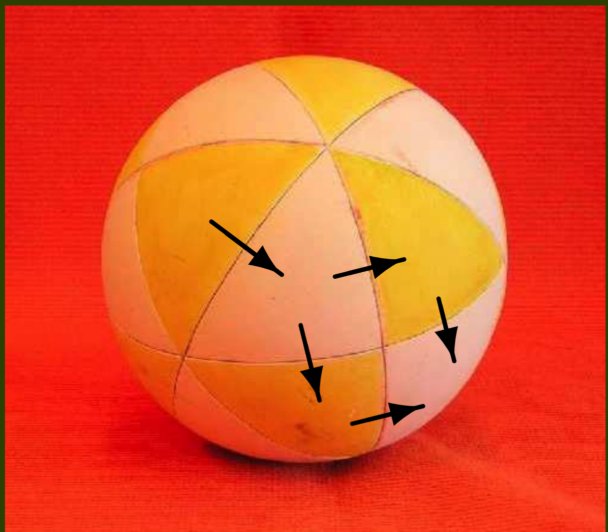
Maxwell's equations are conformally invariant!

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \rightrightarrows \begin{matrix} \Lambda^2_+ \\ \oplus \\ \Lambda^2_- \end{matrix} \rightrightarrows \Lambda^3 \longrightarrow \Lambda^4 \to 0$$

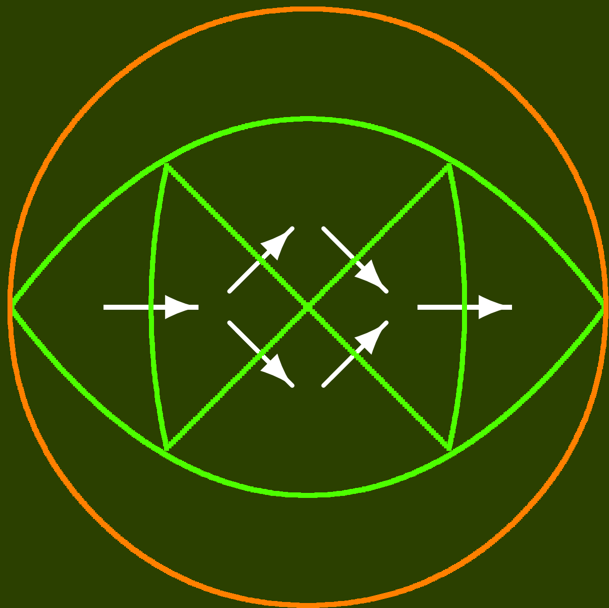
de Rham cont'd

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \to \begin{matrix} \Lambda^2_+ \\ \oplus \\ \Lambda^2_- \end{matrix} \to \Lambda^3 \longrightarrow \Lambda^4 \to 0$$

Road map



http://www.math.rug.nl/models



Can view as
a lune on a sphere!

The countries are
A3 Weyl chambers!

Differential complexes and classification

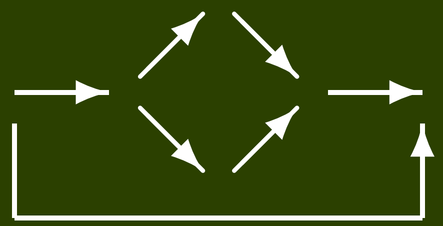
Parabolic subgroups of $\mathrm{SL}(4,\mathbb{R})$

$\stackrel{1:1}{\longleftrightarrow}$ lunes compatible with the A3 tiling of the sphere

Parabolic differential geometry

Parabolic differential geometry cont'd

Enhanced road map



Conformal Laplacian

NB $\mathrm{SL}(4,\mathbb{R})$ also acts on the three-sphere

$\rightsquigarrow$ projective differential geometry

$\rightsquigarrow$ applications in numerical analysis

In the other direction (differential equations $\to$ symmetries)

Laplacian on $\mathbb{R}^n \rightsquigarrow$ conformal symmetries

$\rightsquigarrow \mathfrak{so}(n+1,1) \rightsquigarrow$ Joseph ideal $\subset \mathfrak{U}(\mathfrak{so}(n+1,1))$

$\rightsquigarrow \cdots$

END OF PART ONE

THANK YOU