Complexes of differential operators

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De Rham complex

in $\mathbb{R}^3$

$$f \overset{\text{grad}}{\longmapsto} \nabla_i f \qquad \omega_i \overset{\text{curl}}{\longmapsto} \epsilon_i{}^{jk}\nabla_j\omega_k \qquad \phi_i \overset{\text{div}}{\longmapsto} \nabla^i\phi_i$$

on a smooth manifold

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \to 0$$

Locally exact

$$\Gamma(U,\Lambda^{p-1}) \xrightarrow{d} \Gamma(U,\Lambda^p) \xrightarrow{d} \Gamma(U,\Lambda^{p+1}) \quad \text{is exact} \quad p \geq 1$$

$$\ker : \Gamma(U,\Lambda^0) \xrightarrow{d} \Gamma(U,\Lambda^1) = \mathbb{R}$$

Rumin complex

$\boxed{\text{in } \mathbb{R}^3}$ X, Y vector fields. Suppose $X, Y, Z \equiv [X, Y]$ span.

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant. Let $H \equiv \operatorname{span}\{X, Y\}$.

$\boxed{\text{on a contact manifold}}$ $H \subset TM \quad \leftrightsquigarrow \quad \Lambda^1 \twoheadrightarrow \Lambda^1_H$

$$\begin{array}{ccccccc} 0 & \to & \mathbb{R} & \to & \Lambda^0 & \to & \Lambda^1 \\ & & & & & \searrow & \downarrow \\ & & & & & & \Lambda^1_H \end{array}$$

defines $d_H : \Lambda^0 \to \Lambda^1_H$

s.t. $\mathbb{R} = \ker : \Lambda^0 \xrightarrow{d_H} \Lambda^1_H$

$\boxed{\text{locally}}$ Darboux $\rightsquigarrow$ $[X, Z] = 0 = [Y, Z]$ wlg

$$\left.\begin{array}{rcl} Xf &=& g \\ Yf &=& h \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} XYg - X^2h + Zg &=& 0 \\ YXh - Y^2g - Zh &=& 0 \end{array}\right.$$

conversely? $\boxed{\text{yes!}}$

Rumin complex cont'd

$$
\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \xrightarrow{d} & \Lambda^4 & \xrightarrow{d} & \Lambda^5 \\
\| & & \| & & \| & & \| & & \| & & \| \\
\Lambda^0 & & \Lambda^1_H & & \Lambda^2_H & & \Lambda^3_H & & \Lambda^4_H & & \\
 & & + & \nearrow^{\text{injve}} & + & \nearrow^{\text{isosm}} & + & \nearrow^{\text{surve}} & + & & \\
 & & L & & \Lambda^1_H \otimes L & & \Lambda^2_H \otimes L & & \Lambda^3_H \otimes L & & \Lambda^4_H \otimes L
\end{array}
$$

Diagram chase (spectral sequence) $\rightsquigarrow$

$$
\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H & \xrightarrow{d_H} & \Lambda^2_{H\perp} & \xrightarrow{d_H^{(2)}} & \Lambda^2_{H\perp} \otimes L & \xrightarrow{d_H} & \Lambda^3_H \otimes L & \xrightarrow{d_H} & \Lambda^5 \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
0 \quad 0 \quad 0 & & -2 \quad 1 \quad 0 & & -3 \quad 0 \quad 1 & & -5 \quad 0 \quad 1 & & -6 \quad 1 \quad 0 & & -6 \quad 0 \quad 0 \\
\dim = 1 & & \dim = 4 & & \dim = 5 & & \dim = 5 & & \dim = 4 & & \dim = 1
\end{array}
$$

Engel complex

$\boxed{\text{in } \mathbb{R}^4}$ X, Y vector fields. Suppose

- $X, Y, Z \equiv [X, Y], W \equiv [Y, Z]$ span,
- (Engel $\rightsquigarrow$ wlg) all other commutators vanish.

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant.

$\boxed{\text{on an Engel manifold}}$ $H \subset TM \quad \leftrightsquigarrow \quad \Lambda^1 \twoheadrightarrow \Lambda^1_H$

$$\left.\begin{array}{rcl} Xf & = & g \\ Yf & = & h \end{array}\right\} \Leftrightarrow \left\{ \begin{array}{rcl} XYg - X^2h + Zg & = & 0 \\ Y^2Xh - Y^3g - YZh - Wh & = & 0 \end{array}\right.$$ locally

Complex $\quad 0 \to \mathbb{R} \to \Lambda^0 \to \Lambda^1_H \to \Delta^2 \to \Delta^3 \to \Lambda^4 \to 0$

ranks	1	2	2	2	1

Five variables

$\boxed{\text{in } \mathbb{R}^5}$ X, Y vector fields. Suppose

- $\underbrace{X, Y}, \underbrace{Z \equiv [X, Y]}, \underbrace{W \equiv [Y, Z], V \equiv [X, Z]}$ span.
- ~~(Engel) all other commutators vanish.~~

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant.

$$H \subset TM \quad \leftrightsquigarrow \quad \Lambda^1 \twoheadrightarrow \Lambda^1_H \quad \rightsquigarrow \quad \Lambda^1 = \Lambda^1_H + \Lambda^2_H + \Lambda^1_H \otimes \Lambda^2_H$$

$$\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H & \xrightarrow{d_H^{(3)}} & \odot^2\Lambda^1_H \otimes \Lambda^2_H & \xrightarrow{d_H^{(2)}} & \Delta^3 & \xrightarrow{d_H^{(3)}} & \Delta^4 & \xrightarrow{d_H} & \Lambda^5 \\
\wr & & \wr & & \wr & & \wr & & \wr & & \wr \\
0 \quad 0 & & -2 \quad 1 & & -5 \quad 2 & & -6 \quad 2 & & -6 \quad 1 & & -6 \quad 0 \\
\dim = 1 & & \dim = 2 & & \dim = 3 & & \dim = 3 & & \dim = 2 & & \dim = 1
\end{array}$$

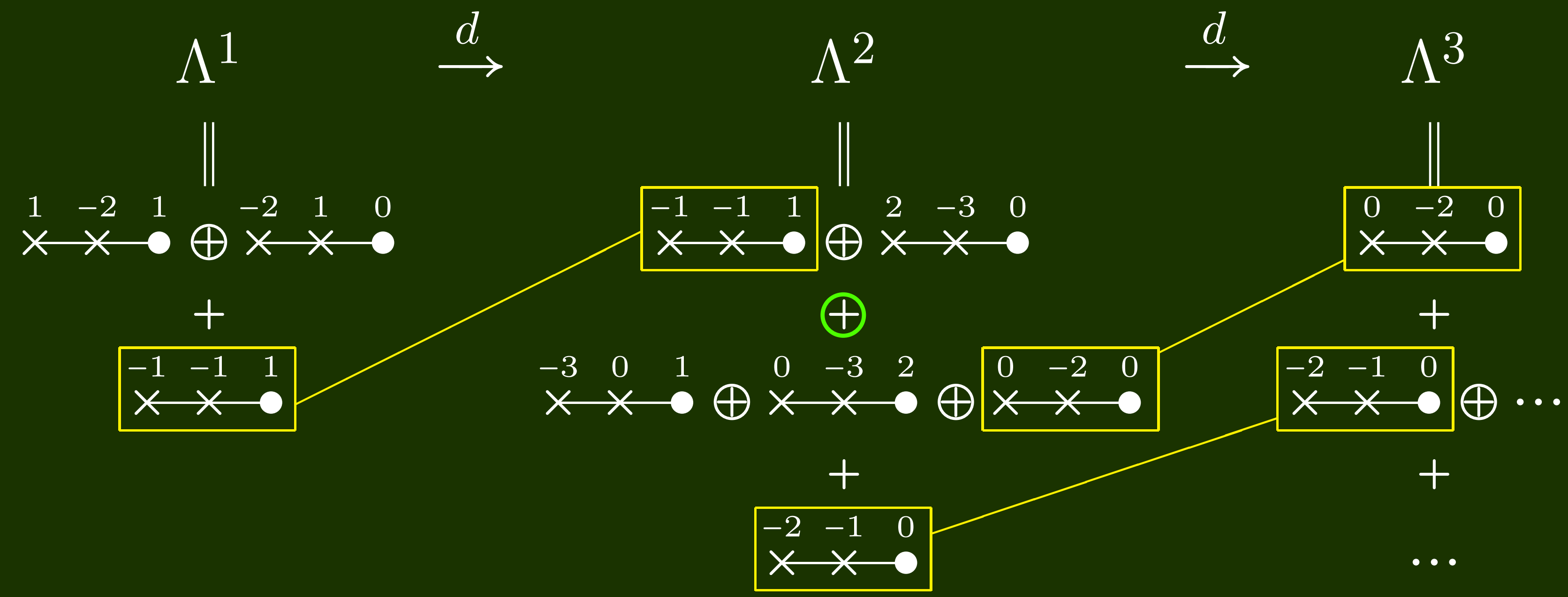
$\dim M = 5, \quad TM \supset H, \quad \operatorname{rank} H = 3.$ Suppose $[H, H] = TM$.

$$
\begin{array}{lcl}
K & \equiv & \ker : \Lambda^1 \to \Lambda^1_H \qquad K \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H \qquad \mathcal{L}(K) \subset \Lambda^2_H \\
D & \subset & H \qquad \text{defined by } \; H = \Lambda^3 H \otimes \Lambda^2_H
\end{array}
$$

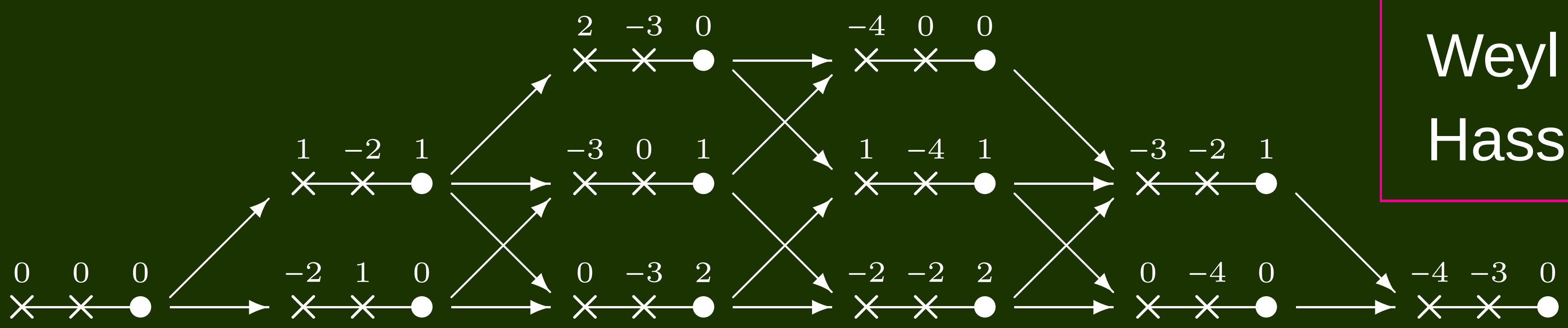
Choose complementary line bundle $H = \xi \oplus D \; (\Leftrightarrow \Lambda^1_H = Q \oplus L)$

$$
\begin{array}{ccccc}
\Lambda^1 & \xrightarrow{\qquad\qquad d \qquad\qquad} & \Lambda^2 & \xrightarrow{\qquad\qquad d \qquad\qquad} & \Lambda^3 \\
\| & & \| & & \| \\
Q \oplus L & & \boxed{Q \otimes L} \oplus \Lambda^2 Q & & \boxed{\Lambda^2 Q \otimes L} \\
+ & \mathcal{L} & + & & + \\
\boxed{Q \otimes L} & & Q \otimes L^2 \oplus \odot^2 Q \otimes L \oplus \boxed{\Lambda^2 Q \otimes L} & & \boxed{\Lambda^2 Q \otimes L^2} \oplus \cdots \\
 & & + & & + \\
 & & \boxed{\Lambda^2 Q \otimes L^2} & & \cdots
\end{array}
$$

××• cont'd



Bernstein-Gelfand-Gelfand complex



Affine action of Weyl group on Hasse diagram

Parabolic geometry

Geometries modelled on homogeneous spaces

$$\boxed{G/P} \qquad \begin{cases} G \text{ simple Lie group} \\ P \text{ parabolic subgroup} \end{cases}$$

Examples

- conformal geometry $\quad \mathrm{SO}(n+1,1)/P$
- CR geometry $\quad \mathrm{SU}(n+1,1)/P$
- projective geometry $\quad \mathrm{SL}(n+1,\mathbf{R})\Big/\left\{\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & * & \cdots & * \end{bmatrix}\right\}$

Chern-Moser connection

I wish to say that I believe that projective differential geometry will be of increasing importance
Shiing-Shen Chern, 1988

Application: linear elasticity

displacement	X_i			
	$\downarrow$ Killing			
strain	$\nabla_i X_j + \nabla_j X_i$	Σ_{ij}		
		$\downarrow$ Saint-Venant		
stress		$\epsilon_i{}^{km}\epsilon_j{}^{\ell n}\nabla_k\nabla_\ell\Sigma_{mn}$	G_{ij}	
			$\downarrow$ Bianchi	
load			$\nabla^i G_{ij}$	

$\boxed{\text{in } \mathbb{R}^3}$

BGG $\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla^{(2)}} \overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet} \xrightarrow{\nabla} \overset{-5}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$ on $\mathbb{RP}_3$

$\rightsquigarrow$ (Arnold-Falk-Winther) new stable finite element schemes

Application: analysis on $\mathbb{CP}_n$

on a symplectic manifold Rumin-Seshadri complex

$$\begin{array}{ccccccccccccc}
\boxed{\Lambda^0} & \xrightarrow{d} & \boxed{\Lambda^1} & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d_\perp} & \Lambda^3_\perp & \xrightarrow{d_\perp} & \cdots & \xrightarrow{d_\perp} & \Lambda^n_\perp \\
 & & & & & & & & & & \downarrow d^{(2)}_\perp \\
\Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda^2_\perp & \xleftarrow{d_\perp} & \Lambda^3_\perp & \xleftarrow{d_\perp} & \cdots & \xleftarrow{d_\perp} & \Lambda^n_\perp
\end{array}$$

$\boxed{}$ local cohomology = $\mathbb{R}$ (cf. Rumin on $S^{2n+1} \to \mathbb{CP}_n$)

- $\Gamma(\mathbb{CP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp)$ exact
- $\Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \odot^2\Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{CP}_n, \boxplus_\perp \Lambda^1)$ exact

Killing cf. St-Venant

THE END

THANK YOU