#1 Regard $S^2$ as the space of oriented lines $\ell$ through the origin $\mathbb{R}^3$. Let $L$ denote the tautological line bundle on $S^2$ that associates to each $\ell$, the unoriented line $\ell$. Show that sections of $L$ over $U^{\text{open}} \subseteq S^2$, may be identified with homogeneous functions $f : \hat{U} \rightarrow \mathbb{R}$ of degree $-1$

$$f(\lambda x) = \lambda^{-1} f(x) \ \forall \ \lambda > 0$$

on $\hat{U}$, the open cone in $\mathbb{R}^3 \setminus \{0\}$ over $U \subseteq S^2$. Show that there is a canonically defined exact sequence of vector bundles on $S^2$:

$$0 \rightarrow L \rightarrow S^2 \times \mathbb{R}^3 \rightarrow TS^2 \otimes L \rightarrow 0,$$

where $TS^2$ is the tangent bundles to $S^2$. Use these observations to construct a non-zero $\text{SL}(3, \mathbb{R})$-invariant linear differential operator (the ‘Killing operator’):

$$\nabla : \Lambda^1(2) \rightarrow \bigwedge^2 \Lambda^1(2).$$

#2 Show that $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ is homogeneous of degree $-2$ if and only if

$$f(u, v)(u\,dv - v\,du)$$

is closed.

#3 Suppose that $\phi : S^2 \rightarrow S^2$ ‘preserves geodesics,’ namely the image $\phi(\gamma)$ of any great circle $\gamma$ is contained in another great circle. Suppose also that $\phi(S^2)$ contains at least 3 non-colinear points (i.e. not all on the same geodesic). Show that if $\phi$ is continuous, then either

(i) $\phi$ is induced by some unique $A \in \text{SL}(3, \mathbb{R})$ acting on $\mathbb{R}^3$,

(ii) $\phi$ is as in (ii) but composed with the antipodal map.

What if $\phi$ is not supposed continuous?

#4 (i) Check that the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ may be realised as differential operators

$$h \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \quad x \mapsto u\frac{\partial}{\partial v} \quad y \mapsto v\frac{\partial}{\partial u}$$

on $\mathbb{R}^2$. Show that this induces a realisation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{R}))$ as differential operators on $\mathbb{R}^2$.

(ii) Check that the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ may be realised as differential operators

$$h \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \quad x_1 \mapsto u\frac{\partial}{\partial v} \quad y_1 \mapsto v\frac{\partial}{\partial u} \quad x_2 \mapsto u\frac{\partial}{\partial w} \quad H \mapsto v\frac{\partial}{\partial w} - w\frac{\partial}{\partial v} \quad X \mapsto v\frac{\partial}{\partial w} \quad Y \mapsto w\frac{\partial}{\partial v} \quad y_2 \mapsto w\frac{\partial}{\partial u}$$

on $\mathbb{R}^3$. Does this induce a realisation of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$ as differential operators on $\mathbb{R}^3$?

#5 (i) Show that $h^2 + 2xy + 2yx$ is in the centre of $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{R}))$.

(ii) Find coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ such that

$$2h^2 + 2H^2 + 2hH + \alpha x_1 y_1 + \beta y_1 x_1 + \gamma XY + \delta YX + \epsilon x_2 y_2 + \zeta y_2 x_2$$

is in the centre of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$.

(ii) Show that

$$2h^3 - 2H^3 + 3h^2 H - 3hH^2 - 18H^2 - 9hH - 18h - 36H - 9H y_2 x_2 + 9h y_2 x_2$$

$$+ 27y_2 x_1 X + 9h y_1 x_1 + 18H y_1 x_1 - 18h YX - 9HYX + 27y_1 Y x_2 - 54Y X - 27y_2 x_2$$

is in the centre of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$.

MikE, Nordfjordeid, 18th June 2015
Notation

\textbf{sl}(2, \mathbb{R}) : \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}

\begin{align*}
\begin{cases}
    h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \quad y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
    H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \quad X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
\end{cases}
\end{align*}

x_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}