QUESTIONS (NOTATION NEXT PAGE)

#1 Regard S^2 as the space of oriented lines $\vec{\ell}$ through the origin $\mathbb{R}^3$. Let L denote the tautological line bundle on S^2 that associates to each $\vec{\ell}$, the unoriented line ℓ. Show that sections of L over $U^{\text{open}} \subseteq S^2$, may be identified with homogeneous functions $f : \tilde{U} \to \mathbb{R}$ of degree -1

$$f(\lambda x) = \lambda^{-1} f(x) \ \ \forall\, \lambda > 0$$

on $\tilde{U}$, the open cone in $\mathbb{R}^3 \setminus \{0\}$ over $U \subseteq S^2$. Show that there is a canonically defined exact sequence of vector bundles on S^2:

$$0 \to L \to S^2 \times \mathbb{R}^3 \to TS^2 \otimes L \to 0,$$

where TS^2 is the tangent bundles to S^2. Use these observations to construct a non-zero $\mathrm{SL}(3, \mathbb{R})$-invariant linear differential operator (the 'Killing operator'):

$$\nabla : \Lambda^1(2) \to \bigodot^2 \Lambda^1(2).$$

#2 Show that $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree -2 if and only if

$$f(u, v)(u\, dv - v\, du) \quad \text{is closed.}$$

#3 Suppose that $\phi : S^2 \to S^2$ 'preserves geodesics,' namely the image $\phi(\gamma)$ of any great circle γ is contained in another great circle. Suppose also that $\phi(S^2)$ contains at least 3 non-colinear points (i.e. not all on the same geodesic). Show that if ϕ is continuous, then either

(i) ϕ is induced by some unique $A \in \mathrm{SL}(3, \mathbb{R})$ acting on $\mathbb{R}^3$,

(ii) ϕ is as in (ii) but composed with the antipodal map.

What if ϕ is not supposed continuous?

#4 (i) Check that the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ may be realised as differential operators

$$h \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \qquad x \mapsto u\frac{\partial}{\partial v} \qquad y \mapsto v\frac{\partial}{\partial u}$$

on $\mathbb{R}^2$. Show that this induces a realisation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{R}))$ as differential operators on $\mathbb{R}^2$.

(ii) Check that the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ may be realised as differential operators

$$\begin{array}{llll} h \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} & x_1 \mapsto u\frac{\partial}{\partial v} & y_1 \mapsto v\frac{\partial}{\partial u} & x_2 \mapsto u\frac{\partial}{\partial w} \\ H \mapsto v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} & X \mapsto v\frac{\partial}{\partial w} & Y \mapsto w\frac{\partial}{\partial v} & y_2 \mapsto w\frac{\partial}{\partial u} \end{array}$$

on $\mathbb{R}^3$. Does this induce a realisation of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$ as differential operators on $\mathbb{R}^3$?

#5 (i) Show that $h^2 + 2xy + 2yx$ is in the centre of $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{R}))$.

(ii) Find coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ such that

$$2h^2 + 2H^2 + 2hH + \alpha x_1 y_1 + \beta y_1 x_1 + \gamma XY + \delta YX + \epsilon x_2 y_2 + \zeta y_2 x_2$$

is in the centre of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$.

(ii) Show that

$$\begin{aligned} &2h^3 - 2H^3 + 3h^2H - 3hH^2 - 18H^2 - 9hH - 18h - 36H - 9Hy_2x_2 + 9hy_2x_2 \\ &+ 27y_2x_1X + 9hy_1x_1 + 18Hy_1x_1 - 18hYX - 9HYX + 27y_1Yx_2 - 54YX - 27y_2x_2 \end{aligned}$$

is in the centre of $\mathfrak{U}(\mathfrak{sl}(3, \mathbb{R}))$.

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Notation

$$\mathfrak{sl}(2,\mathbb{R}): \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathfrak{sl}(3,\mathbb{R}): \quad \begin{cases} h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \qquad x_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{cases}$$