Twistors, tractors, and conformally invariant operators

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Twistors

What's a twistor? Twistor equation $\nabla_{A'(A}\omega_{B)} = 0$

Recall $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab} \quad \rightsquigarrow \quad \begin{array}{l} \epsilon_{AB} \mapsto \widehat{\epsilon}_{AB} = \Omega\epsilon_{AB} \\ \epsilon_{A'B'} \mapsto \widehat{\epsilon}_{A'B'} = \Omega\epsilon_{A'B'} \end{array}$

$$\widehat{\nabla}_{A'A}\omega_B = \nabla_{A'A}\omega_B + w\Upsilon_{A'A}\omega_B - \Upsilon_{A'B}\omega_A$$

$w = 1$ $\quad \widehat{\nabla}_{A'A}\omega_B = \nabla_{A'A}\omega_B + 2\Upsilon_{A'[A}\omega_{B]} \Rightarrow \widehat{\nabla}_{A'(A}\omega_{B)} = \nabla_{A'(A}\omega_{B)}$

Prolong $\quad \nabla_{A'(A}\omega_{B)} = 0 \quad \Longleftrightarrow \quad \nabla_{AA'}\omega_B = \epsilon_{AB}\pi_{A'}$

$$\Longleftrightarrow \quad \begin{array}{l} \nabla_{AA'}\omega_B = \epsilon_{AB}\pi_{A'} \\ \nabla_{AA'}\pi_{B'} = \mathrm{P}_{A'B'A}{}^{B}\omega_B \end{array}$$

Local twistor transport

Local twistor transport

$$\begin{bmatrix} \omega_B \\ \pi_{B'} \end{bmatrix} \xmapsto{\nabla_a} \begin{bmatrix} \nabla_{AA'}\omega_B - \epsilon_{AB}\pi_{A'} \\ \nabla_{AA'}\pi_{B'} + \mathrm{P}_{A'B'AB}\omega^B \end{bmatrix}$$ connection

$$0 \longrightarrow \mathcal{E}_{B'} \longrightarrow \mathcal{E}^\beta \longrightarrow \mathcal{E}_B[1] \longrightarrow 0$$

conformally invariant operators

$$\mathcal{E}_B[1] \xrightarrow{\nabla_a} \mathcal{E}_{A'(AB)}[1]$$ twistor operator

$$\mathcal{E}^\beta \xrightarrow{\nabla_a} \mathcal{E}_{AA'}{}^\beta$$ twistor connection

} combine

$$\begin{matrix} \mathcal{E}_{BC}[2] \\ + \\ \mathcal{E}_{BC'}[1] \end{matrix} = \mathcal{E}_B{}^\gamma[1] \xrightarrow{\nabla_a} \mathcal{E}_{A'(AB)}{}^\gamma[1] = \begin{matrix} \mathcal{E}_{A'(AB)C}[2] \\ + \\ \mathcal{E}_{(AB)A'C'}[1] \end{matrix}$$

Translation: new operators from old

$$\mathcal{E}_{(BC)}[2] \oplus \mathcal{E}[1] = \mathcal{E}_{BC}[2] \qquad \mathcal{E}_{A'(AB)C}[2] = \mathcal{E}_{A'(ABC)}[2] \oplus \mathcal{E}_{A'A}[1]$$

$$+ \qquad \longrightarrow \qquad + \qquad \simeq$$

$$\mathcal{E}_{BC'}[1] \qquad \mathcal{E}_{(AB)A'C'}[1] = \mathcal{E}_{(AB)(A'C')}[1] \oplus \mathcal{E}_{(AB)}$$

- $\mathcal{E}_{(BC)}[2] \ni \phi_{BC} \longmapsto \nabla_{A'(A}\phi_{BC)} \in \mathcal{E}_{A'(ABC)}[2]$
  higher twistor operator
- $\mathcal{E}[1] \ni \sigma \longmapsto \left(\nabla_{(A}{}^{(A'}\nabla_{B)}{}^{B')} + \mathrm{P}_{(AB)}{}^{(A'B')}\right)\sigma \in \mathcal{E}_{(AB)}{}^{(A'B')}[-1]$
  Conformal-to-Einstein operator LeBrun 1985

$$\mathcal{E}[1] \ni \sigma \mapsto (\nabla_a\nabla_b + \mathrm{P}_{ab})_\circ\sigma \in \mathcal{E}_{(ab)\circ}[1]$$

$(\nabla_a\nabla_b + \mathrm{P}_{ab})_\circ\sigma = 0 \iff \sigma^{-2}g_{ab}$ is Einstein !

Tractors

Almost Einstein $(\nabla_a\nabla_b + \mathrm{P}_{ab})_\circ\sigma = 0$ Prolong

$$(\nabla_a\nabla_b + \mathrm{P}_{ab})_\circ\sigma = 0 \iff \begin{array}{l} \nabla_a\sigma = \mu_a \\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho = 0 \end{array}$$

$$\iff \boxed{\begin{array}{l} \nabla_a\sigma - \mu_a = 0 \\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho = 0 \\ \nabla_a\rho - \mathrm{P}_a{}^b\mu_b = 0 \end{array}}$$

Tractor transport

$2\sigma\rho + \mu^b\mu_b$

$\mathrm{SO}(n+1,1)$-connection

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \stackrel{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho \\ \nabla_a\rho - \mathrm{P}_a{}^b\mu_b \end{bmatrix}$$

Cartan 1923

Thomas 1926

Twistors versus tractors

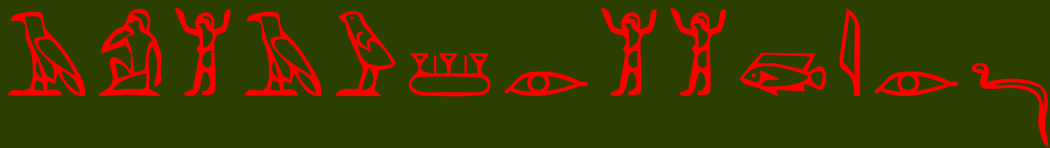
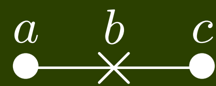
Tracey Thomas
conformal geometry

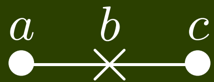
vector	tensor	spinor
	tractor*	twistor

*Hodges 1991

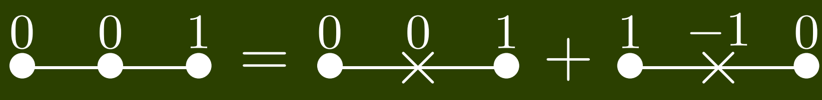
Twistors $\mathcal{E}^{\alpha} = \mathcal{E}_A[1] + \mathcal{E}_{A'}$ Tractors $\mathcal{E}^{[\alpha\beta]} = \mathcal{E}[1] + \mathcal{E}_a[1] + \mathcal{E}[-1]$

Connections agree! Warning

Hieroglyphics  



$= \mathcal{E}_{\underbrace{A'B'\cdots C'}_{a}\underbrace{DE\cdots F}_{c}}[a+b+c]$

Twistors 

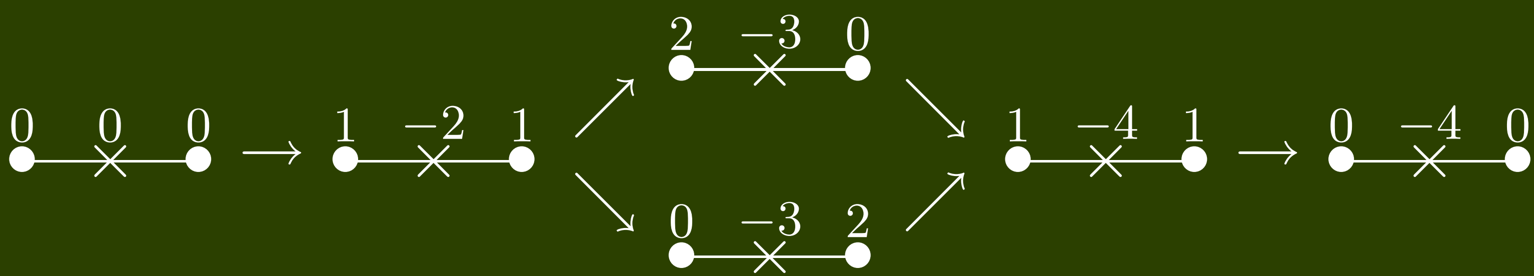
Tractors 

de Rham

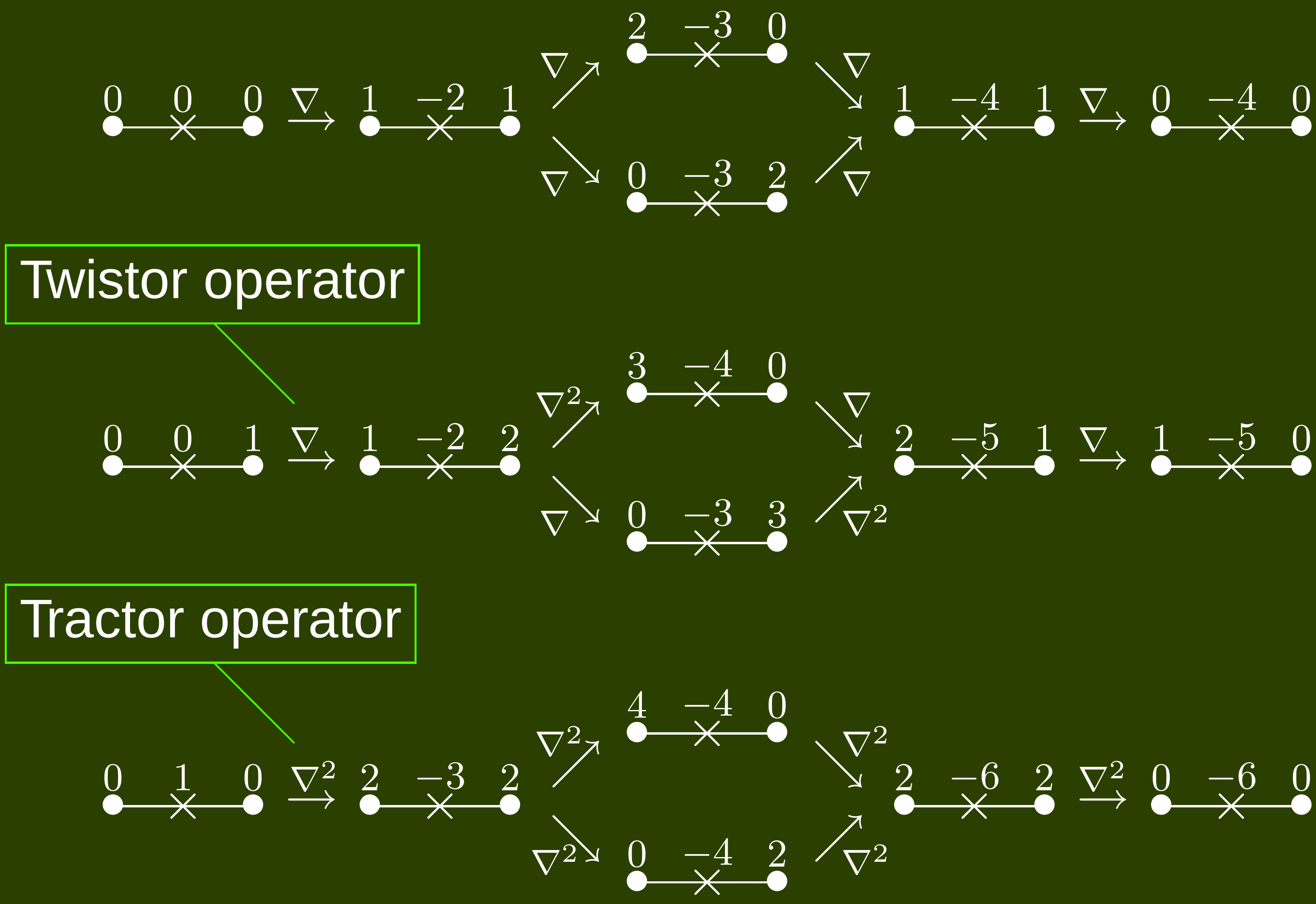
The de Rham complex in four (conformal) variables

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \to \begin{array}{c}\Lambda^2_+ \\ \oplus \\ \Lambda^2_-\end{array} \to \Lambda^3 \longrightarrow \Lambda^4 \to 0$$

$$\mathcal{E} \to \mathcal{E}_{AA'} \to \begin{array}{c}\mathcal{E}_{(A'B')}[-1] \\ \\ \mathcal{E}_{(AB)}[-1]\end{array} \to \mathcal{E}_{AA'}[-2] \to \mathcal{E}[-4]$$

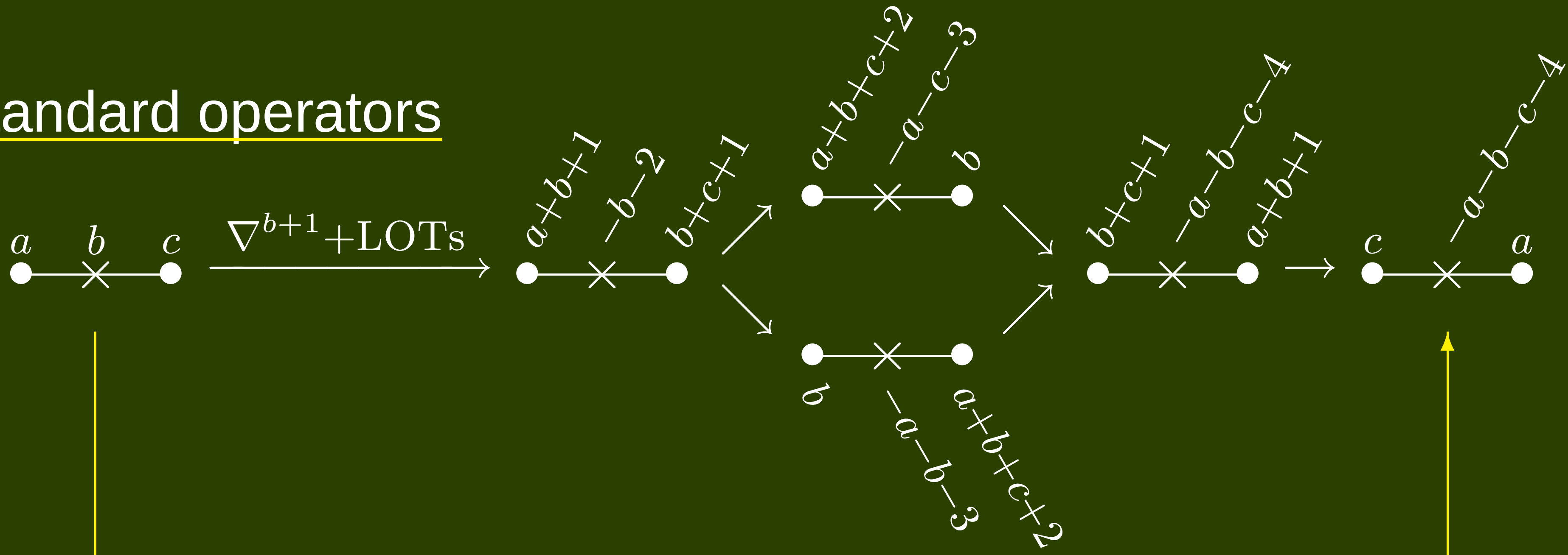


Bernstein-Gelfand-Gelfand = BGG



Classification

Standard operators



Non-standard operators

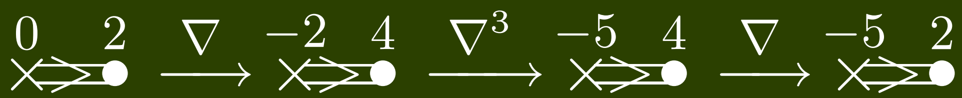
under very special circumstances

$a = c = 0 \rightsquigarrow$ Yamabe ($b = -1$) Paneitz ($b = 0$)

Proof Translation using twistors and/or tractors

Examples in three dimensions

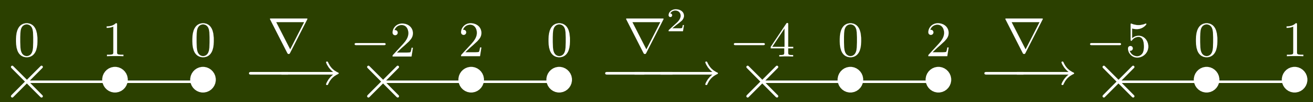
Conformal BGG (Conformal Killing operator et cetera)



$$X_a \mapsto (\nabla_{(a}X_{b)})_\circ \qquad \sigma^{ab} \mapsto \nabla_a\sigma^{ab}$$

$$\phi_{ab} \mapsto (\nabla^{(a}\epsilon^{b)cd}\nabla_c\nabla^e\phi_{de} + \mathrm{LOTs})_\circ$$

Projective BGG (Killing operator et cetera)



$$X_a \mapsto \nabla_{(a}X_{b)} \qquad \sigma^{ab} \mapsto \nabla_a\sigma^{ab}$$

$$\phi_{ab} \mapsto \epsilon^{ace}\epsilon^{bdf}\nabla_c\nabla_d\phi_{ef} + \mathrm{LOTs}$$

THE END

THANK YOU