Bernstein-Gelfand-Gelfand Complexes

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Stress and strain in 3-space

Killing

Saint-Venant

Bianchi

displacement X_i

$\downarrow$

strain $\nabla_i X_j + \nabla_j X_i \qquad \Sigma_{ij}$

$\downarrow$

stress $\epsilon^{ikm}\epsilon^{j\ell n}\nabla_k \nabla_\ell \Sigma_{mn} \qquad G^{ij}$

$\downarrow$

load $\nabla_i G^{ij}$

cf. de Rham

$$f \xmapsto{\text{grad}} \nabla_i f \qquad \omega_i \xmapsto{\text{curl}} \epsilon^{ijk}\nabla_j \omega_k \qquad \phi^i \xmapsto{\text{div}} \nabla_i \phi^i$$

$$\boxed{\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} T \xrightarrow{d} \Lambda^0}$$

A flat connection

$$\begin{pmatrix} X_j \\ Y^j \end{pmatrix} \mapsto \nabla_i \begin{pmatrix} X_j \\ Y^j \end{pmatrix} \equiv \begin{pmatrix} \nabla_i X_j + \epsilon_{ijk} Y^k \\ \nabla_i Y^j \end{pmatrix}$$

Compute:–

$$\begin{aligned} \nabla_i \nabla_j \begin{pmatrix} X_k \\ Y^k \end{pmatrix} &= \nabla_i \begin{pmatrix} \nabla_j X_k + \epsilon_{jk\ell} Y^\ell \\ \nabla_j Y^k \end{pmatrix} \\ &= \begin{pmatrix} \nabla_i(\nabla_j X_k + \epsilon_{jk\ell} Y^\ell) + \epsilon_{ik\ell} \nabla_j Y^\ell \\ \nabla_i \nabla_j Y^k \end{pmatrix} \\ &= \begin{pmatrix} \nabla_i \nabla_j X_k + \epsilon_{jk\ell} \nabla_i Y^\ell + \epsilon_{ik\ell} \nabla_j Y^\ell \\ \nabla_i \nabla_j Y^k \end{pmatrix} \end{aligned}$$

is symmetric in ij !!!

Coupled de Rham

$$\begin{pmatrix} X_k \\ Y^k \end{pmatrix} \overset{\text{grad}}{\longmapsto} \begin{pmatrix} \nabla_j X_k + \epsilon_{jk\ell} Y^\ell \\ \nabla_j Y^k \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} P_{jk} \\ Q_j{}^k \end{pmatrix} \overset{\text{curl}}{\longmapsto} \epsilon^{ijm} \begin{pmatrix} \nabla_i P_{jk} + \epsilon_{ik\ell} Q_j{}^\ell \\ \nabla_i Q_{jk\ell} \end{pmatrix}$$

$$\begin{array}{ccccccc} \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & T \otimes \Lambda^1 & & \Lambda^1 \\ \oplus & \xrightarrow{\text{grad}} & \oplus & \xrightarrow{\text{curl}} & \oplus & \xrightarrow{\text{div}} & \oplus \\ T & & \Lambda^1 \otimes T & \cong & T \otimes T & & T \end{array}$$

FEEC

$$\begin{pmatrix} 0 \\ Q_j{}^k \end{pmatrix} \longmapsto \epsilon^{ijm} \begin{pmatrix} \epsilon_{ik\ell} Q_j{}^\ell \\ * \end{pmatrix} = \begin{pmatrix} Q_k{}^m - Q_\ell{}^\ell \delta_k{}^m \\ * \end{pmatrix}$$

Coupled de Rham after cancellation

$$\begin{array}{ccccccc} \Lambda^1 & \xrightarrow{\nabla} & \Lambda^1 \odot \Lambda^1 & \xrightarrow{\nabla^2} & T \odot T & \xrightarrow{\nabla} & T \\ X_i & & \Sigma_{ij} & & G^{ij} & & B^j \end{array}$$

Linear elasticity complex = A BGG complex

BGG-FEEC

Other BGG complexes

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 \odot_\circ \Lambda^1 \xrightarrow{\nabla^3} T \odot_\circ T \xrightarrow{\nabla} T$$

= linearised conformal deformation complex

$$\begin{array}{ccccccc} \Lambda^0 & \xrightarrow{\nabla^2} & \Lambda^1 \odot \Lambda^1 & \xrightarrow{\nabla} & T \otimes_\circ \Lambda^1 & \xrightarrow{\nabla} & \Lambda^1 \\ f & \xmapsto{\text{Hess}} & \nabla_i \nabla_j f & & & & \end{array}$$

ranks $1 \;\; 6 \;\; 8 \;\; 3$

Further reading

- D.N. Arnold, R.S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica 15 (2006) 1–155.
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THANK YOU