1. Sylvester Canonical Form

In [1], J.J. Sylvester showed that a generic binary sextic Q could be thrown into what is now called 'Sylvester Canonical Form' under the action of $\mathrm{SL}(2,\mathbb{C})$. We take the opportunity here to explore his proof with a view to determining exactly the sextics that are excluded by his genericity assumptions. Remarkably, it turns out that, up to scale, there are precisely 7 sextics omitted from his canonical form.

Theorem 1. *A generic binary sextic may effectively be thrown into Sylvester Canonical Form*

$$Au^6 + Bv^6 + Cw^6 + Duvw(u-v)(v-w)(w-u) \tag{1}$$

for linear forms u, v, w satisfying $u+v+w=0$ (any two of which we may suppose to be homogeneous coördinates on $\mathbb{CP}_1$). The following 7 exceptional sextics

(a) x^4y^2
(b) $x^4(x^2+y^2)$
(c) x^5y
(d) $(x^5+y^5)y$
(e) x^3y^3
(f) $2x^6 + 18x^5y + 10x^3y^3 - y^6$
(g) $186x^6 - 192x^5y - 300x^4y^2 - 320x^3y^3 - 150x^2y^4 - 48xy^5 + 23y^6$

cannot be placed into this form. Up to the action of $\mathrm{GL}(2,\mathbb{C})$*, this is a complete list of all binary sextics.*

Proof. The general binary sextic

$$Q = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6$$

transforms as a symmetric tensor a_{ijklmn} under $\mathrm{SL}(2,\mathbb{C})$ where

$$a_{111111} = a \quad a_{111112} = b \quad a_{111122} = c \quad a_{111222} = d \quad \cdots \quad a_{222222} = g.$$

As such, it may be equivalently regarded as an endomorphism

$$c_{ijk} \overset{\widehat{Q}}{\longmapsto} a_{ijklmn}\epsilon^{lp}\epsilon^{mq}\epsilon^{nr}c_{pqr}$$

on the space of binary cubics. In terms of matrices, we obtain

$$\begin{pmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix} \overset{\widehat{Q}}{\longmapsto} \begin{pmatrix} -d & -3e & -3f & -g \\ c & 3d & 3e & f \\ -b & -3c & -3d & -e \\ a & 3b & 3c & d \end{pmatrix} \begin{pmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{pmatrix} \tag{2}$$

and, computing the characteristic polynomial of this matrix, we find

$$\lambda^4 + (ag - 6bf + 15ce - 10d^2)\lambda^2 + 9(d^4 - 3cd^2e + \cdots + aceg),$$

a double quadratic. It may be solved explicitly to find the 'eigencubics' of Q, namely those binary cubics c_{ijk} such that

$$(\widehat{Q}c)_{ijk} = \lambda c_{ijk}$$

for some $\lambda \in \mathbb{C}$. Suppose that one of these cubics has distinct roots. Absorbing any overall constant into λ, we may choose homogeneous coördinates $[X, Y]$ so that this eigencubic has the form $X^3 + Y^3$. We have thereby arranged that $\widehat{Q}(X^3 + Y^3) = \lambda(X^3 + Y^3)$ but notice that

$$\widehat{X^6 - Y^6}(X^3 + Y^3) = (X^3 + Y^3).$$

We conclude that, in these new coördinates,

$$P \equiv Q - \lambda(X^6 - Y^6) \implies \widehat{P}(X^3 + Y^3) = 0.$$

From (2) we conclude that

$$P = A(X + Y)^6 + B(\omega X + \omega^2 Y)^6 + C(\omega^2 X + \omega Y)^6,$$

for suitable constants A, B, C, where $\omega = e^{2\pi i/3}$. Setting

$$u \equiv X + Y, \quad v \equiv \omega X + \omega^2 Y, \quad w \equiv \omega^2 X + \omega Y$$

and reassembling the result gives Sylvester's Canonical Form (1).

Now we must deal with the case when none of the eigencubics of Q has distinct roots, this being the only issue that arose in throwing a sextic into the form (1). Firstly, we shall check that the 7 claimed exceptional sextics listed in the statement of this theorem lie outside the realm of Sylvester Canonical Form and are mutually distinct under the action of $\mathrm{GL}(2, \mathbb{C})$. For the moment, let us replace the final sextic in the list by

$$\text{(g}'\text{)} \qquad x^6 + 18x^5y + 40(1 - 2\sqrt{7})x^3y^3 + 32(115 - 41\sqrt{7})y^6.$$

It is straightforward to check that the only eigencubics for these seven sextics are as follows.

sextic	basis of eigencubics
(1)	x^3
(2)	x^3
(3)	$x^3,\ x^2y$
(4)	x^2y
(5)	$x^3,\ x^2y,\ xy^2,\ y^3$
(6)	$x^2y,\ (2x - y)(x + y)^2$
(7′)	$x^2y,\ (x - y - \sqrt{7}y)(x - 4y + 2\sqrt{7}y)^2,$ $(2x - 2y \pm 3iy + \sqrt{7}y)(x + 2y - \sqrt{7}y \pm 3iy)^2$

Notice that none of these eigencubics has a repeated root. Furthermore, apart from (1) and (2), the arrangement of the consequent triple and double roots cleanly distinguishes between these cases whilst (1) and (2) are distinguished from each other by the arrangement of roots of the sextics themselves (e.g., only (1) has a doule root). We conclude that (1)–(7) cannot be

CHECK CAREFULLY AND DRAW CONCLUSIONS

INTRO... There are two cases

- there is an eigencubic with a double root,
- there is an eigencubic with a triple root.

Without loss of generality, we may change coördinates such that the resulting eigencubic has the form x^2y or x^3. In case it is x^3, we can read off from (2) that the sextic is necessarily of the form

$$Q = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3$$

and conclude that

$$\widehat{Q}x^3 = -dx^3 \quad \text{and} \quad \widehat{Q}x^2y = cx^3 + 3dx^2y.$$

Therefore,

$$\widehat{Q}x^2(cx + 4dy) = 3dx^2(cx + 4dy)$$

and $x^2(cx + 4dy)$ is an eigencubic. If $d \neq 0$, then this is an eigencubic with a double root. If $c = d = 0$, then x^2y is an eigencubic. Therefore, in case of a triple root, either we may reduce to the case a double root or we may rescale Q to be of the form

$$Q = ax^6 + 6bx^5y + x^4y^2$$

at which point the substitution $y \mapsto y - 3bx$ eliminates the term in x^5y. If $a = 0$, we are led to x^4y^2. Otherwise, the substitution $x \mapsto x/\sqrt{a}$ and $Q \mapsto a^2Q$ give us the canonical form $x^4(x^2 + y^2)$. We have obtained the first two of the exceptional cases.

In all other cases $\widehat{Q}$ has an eigencubic with double root, in which case we may as well take it to be x^2y. This forces $c = e = f = 0$ in (2). The resulting matrix

$$(3) \qquad \begin{pmatrix} -d & 0 & 0 & -g \\ 0 & 3d & 0 & 0 \\ -b & 0 & -3d & 0 \\ a & 3b & 0 & d \end{pmatrix}$$

has characteristic polynomial $(\lambda + 3d)(\lambda - 3d)(\lambda^2 + ag - d^2)$ and we may explicitly find all eigencubics. Apart from x^2y with eigenvalue $3d$, we note that

$$(\widehat{Q} + 3d\,\mathrm{Id})(8bd^2x^3 - b^2gx^2y + 2d(ag + 8d^2)xy^2 + 2bdgy^3) = 0,$$

potentially providing the eigencubic for eigenvalue $-3d$ and also that

$$(\widehat{Q}-\lambda\,\mathrm{Id})((d-\lambda)(3d-\lambda)x^3-3bgx^2y+(3d-\lambda)gy^3)=(3d-\lambda)(\lambda^2+ag-d^2)x^3$$

potentially provides the eigencubics for $\lambda=\pm\sqrt{d^2-ag}$. In case these eigencubics are linearly independent, we have diagonalised (3). To see whether this is the case, we compute the determinant

$$\left|\begin{array}{cccc} 0 & 8bd^2 & (d-\lambda)(3d-\lambda) & (d+\lambda)(3d+\lambda) \\ 1 & -b^2g & -3bg & -3bg \\ 0 & 2d(ag+8d^2) & 0 & 0 \\ 0 & 2bdg & (3d-\lambda)g & (3d+\lambda)g \end{array}\right|,$$

obtaining $4d(8d^2+ag)\lambda(\lambda-3d)g(3d+\lambda)$. Therefore, our first task is to investigate what happens if any of the factors in this determinant vanish.

Suppose $d=0$ Then

Recall that if any of these any eigencubics has distinct roots, then the corresponding sextic may be thrown into Sylvester Canonical Form. Therefore, we should determine the sextics for which all these eigencubics have only multiple roots. In other words, we should consider the discriminants of the eigencubics (other than x^2y) as follows

$$\begin{array}{rcl} \mathrm{Dis}_{-3d} & \equiv & \mathrm{discrim}(8bd^2x^3-b^2gx^2y+2d(ag+8d^2)xy^2+2bdgy^3) \\ & = & 4bd\left(\begin{array}{c} a^2b^3dg^4-128ab^3d^3g^3-32768d^{10}-1536a^2d^6g^2 \\ +2b^6g^4-64a^3d^4g^3-2816b^3d^5g^2-12288ad^8g \end{array}\right) \\ \mathrm{Dis}_{\lambda} & \equiv & \mathrm{discrim}((d-\lambda)(3d-\lambda)x^3-3bgx^2y+(3d-\lambda)y^3) \\ & = & 27g^2(3d-\lambda)\left(\begin{array}{l} 4b^3g^2-27d^5+81d^4\lambda \\ -90d^3\lambda^2+46d^2\lambda^3-11d\lambda^4+\lambda^5 \end{array}\right) \\ \mathrm{Dis}_{-\lambda} & \equiv & \mathrm{discrim}((d+\lambda)(3d+\lambda)x^3-3bgx^2y+(3d+\lambda)y^3) \\ & = & 27g^2(3d+\lambda)\left(\begin{array}{l} 4b^3g^2-27d^5-81d^4\lambda \\ -90d^3\lambda^2-46d^2\lambda^3-11d\lambda^4-\lambda^5 \end{array}\right) \end{array}$$

The remaining sextics that cannot be thrown into Sylvester Canonical Form are, therefore, parameterised by solutions to the following system of equations

$$\{\mathrm{Dis}_{-3d}=0,\mathrm{Dis}_{\lambda}=0,\mathrm{Dis}_{-\lambda}=0,\lambda^2=d^2-ag\}.$$

Remarkably, this system of equations may be solved explicitly.

DO IT! $\square$

References

[1] J.J. Sylvester, *On the calculus of forms, otherwise the theory of invariants*, Cambridge and Dublin Mathematical Journal **IX** (1854), 85–104.