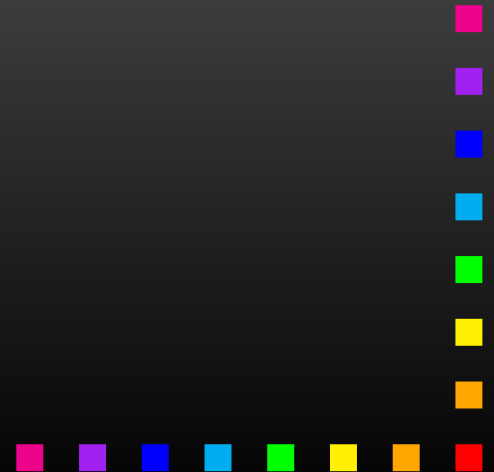


Re-inventing the Wheel: Differential Operators on the Sphere

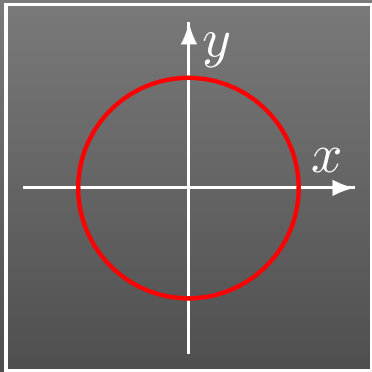
Michael Eastwood

Australian National University



The round wheel

The round circle: $\{x^2 + y^2 = 1\} = \{(\cos \theta, \sin \theta)\}$

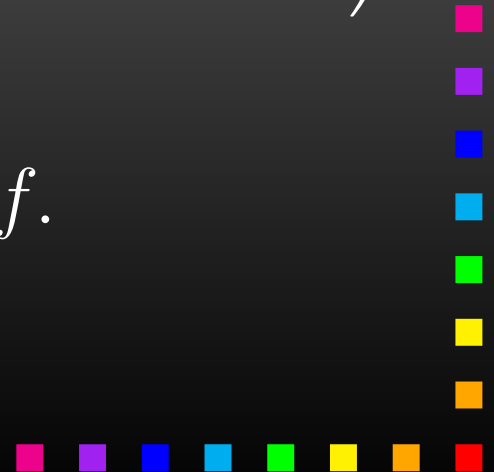


with symmetry group $SO(2)$

Invariant differential operators:—
$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

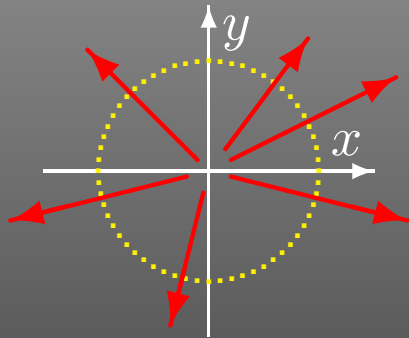
$$f \mapsto 2 \frac{d^3 f}{d\theta^3} + 7 \frac{d^2 f}{d\theta^2} - \pi^{42} \frac{df}{d\theta} - f.$$

Any polynomial in $d/d\theta$ will do.



The projective wheel

The projective circle: $\{(x, y) \sim \lambda(x, y) \text{ for } \lambda > 0\}$



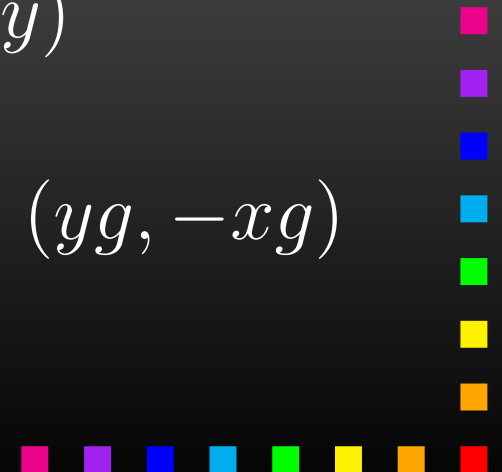
with symmetry group $SL(2, \mathbb{R})$.

Homogeneous space: $SL(2, \mathbb{R}) / \left\{ \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \right\}$.

Function on circle $\Leftrightarrow f(\lambda x, \lambda y) = f(x, y)$

$$\Leftrightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0 \quad \Leftrightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (yg, -xg)$$

where $g(\lambda x, \lambda y) = \lambda^{-2}g(x, y)$.



An invariant operator

Claim: $f \mapsto g \equiv \frac{1}{y} \frac{\partial f}{\partial x} = -\frac{1}{x} \frac{\partial f}{\partial y}$ is $SL(2, \mathbb{R})$ -invariant.

Proof: Fix skew tensor ϵ_{ij} (volume form). Then

$$\nabla_i f = \epsilon_{ij} x^j g \quad \text{defines } g. \quad \square$$

Write $g = \nabla f$ for

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E}(-2) = \Lambda^1.$$

exterior derivative

Homogeneous line bundles on $SL(2, \mathbb{R})/P$



More invariant operators

Claim: there are $SL(2, \mathbb{R})$ -invariant operators

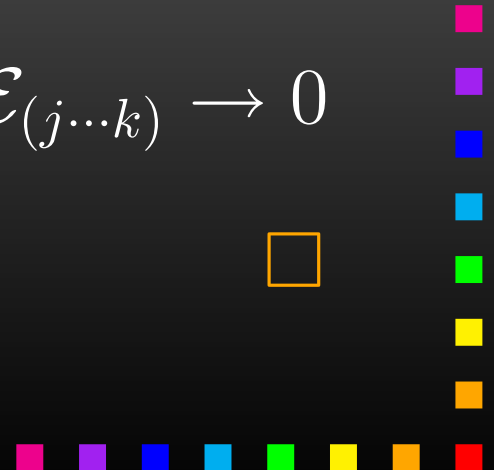
$$\nabla^{(\ell+1)} : \mathcal{E}(\ell) \longrightarrow \mathcal{E}(-\ell - 2) \quad \forall \ell \geq 0.$$

Proof: $f_{ij\dots k} \equiv \nabla_i \nabla_j \cdots \nabla_k f$ ($\ell + 1$ derivatives)

- $f_{ij\dots k} = f_{(ij\dots k)}$ and $x^i f_{ij\dots k} = 0$
- exact sequence

$$0 \longrightarrow \mathcal{E}(-\ell - 2) \longrightarrow \mathcal{E}_{(ij\dots k)}(-1) \xrightarrow{x^i} \mathcal{E}_{(j\dots k)} \longrightarrow 0$$

$$g \longmapsto \epsilon_{ip} \epsilon_{jq} \cdots \epsilon_{kr} x^p x^q \cdots x^r g. \quad \square$$



Classification!

Theorem We've found all $SL(2, \mathbb{R})$ -invariant linear differential operators on the circle: if $D : \mathcal{E}(\ell) \rightarrow \mathcal{E}(m)$ is a non-constant invariant linear differential operator, then

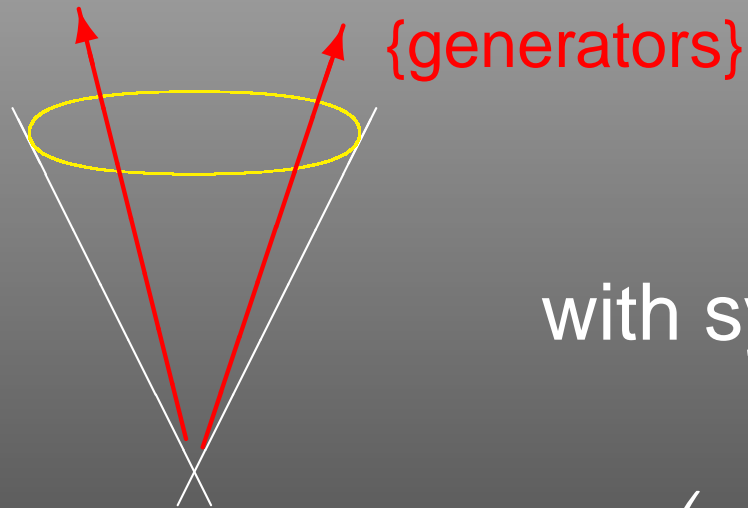
- $\ell \geq 0$ and $m = -\ell - 2$
- $D = \text{constant} \times \nabla^{(\ell+1)}$.

Best proof \leftrightarrow representation theory of $\mathfrak{sl}(2, \mathbb{R})$.

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \dots$$

$$[x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y \quad \&c.$$

Another wheel



with symmetry group $SO(2, 1)$.

But, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$ induces

$$SL(2, \mathbb{R}) \xrightarrow{\simeq 2:1} SO(2, 1)$$

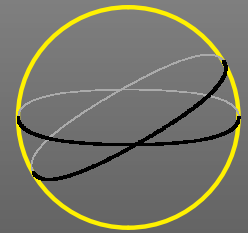
$$\rightsquigarrow (SL(2, \mathbb{R})/P)/\pm \text{Id} \quad (\text{not much different}).$$



The 2-sphere

Various symmetry groups:–

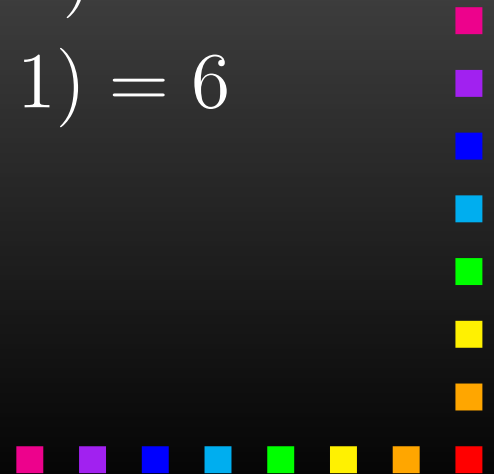
- The round sphere = $SO(3)/SO(2)$
- The projective sphere = $SL(3, \mathbb{R})/P$
- The conformal sphere = $SO(3, 1)/P$
- The Riemann sphere = $SL(2, \mathbb{C})/P$



Projective \neq conformal $\left\{ \begin{array}{l} \dim_{\mathbb{R}} SL(3, \mathbb{R}) = 8 \\ \dim_{\mathbb{R}} SO(3, 1) = 6 \end{array} \right.$

Conformal = Riemann

$$SL(2, \mathbb{C}) \xrightarrow{\cong 2:1} SO(3, 1).$$



The 3-sphere

Many symmetry groups, including

- The projective 3-sphere = $SL(4, \mathbb{R})/P$
- The conformal 3-sphere = $SO(4, 1)/P$
- The contact projective 3-sphere = $Sp(4, \mathbb{R})/P$
- The CR 3-sphere = $SU(2, 1)/P$

They are all different:—



Also as **complex** homogeneous spaces.

$S^3 = SU(2) \Rightarrow$ many more possibilities...



SL(4, ℝ)-invariant operators

- $f \in \Gamma(S^3, \mathcal{E}(0)) \Rightarrow x^i \nabla_i f = 0$. But...

$$0 \rightarrow \Lambda^1 \rightarrow \mathcal{E}_i(-1) \xrightarrow{x^i} \mathcal{E} \rightarrow 0 \quad \text{Euler sequence}$$

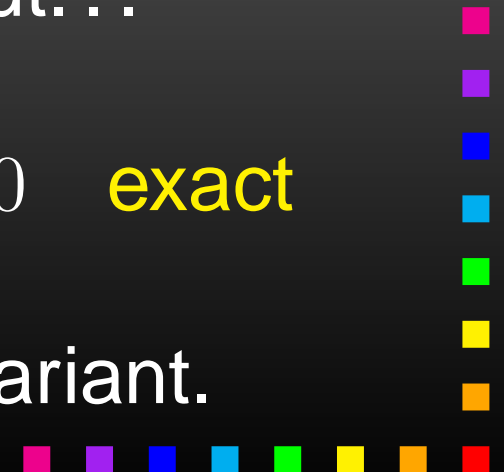
bundle of 1-forms on S^3

and $\nabla : \Lambda^0 \rightarrow \Lambda^1$ is invariant (exterior derivative).

- $f \in \Gamma(S^3, \mathcal{E}(1)) \Rightarrow x^i \nabla_i \nabla_j f = 0$. But...

$$0 \rightarrow \odot^2 \Lambda^1(1) \rightarrow \mathcal{E}_{(ij)}(-1) \xrightarrow{x^i} \mathcal{E}_j \rightarrow 0 \quad \text{exact}$$

and so $\nabla^{(2)} : \Lambda^0(1) \rightarrow \odot^2 \Lambda^1(1)$ is invariant.



More...

- $f \in \Gamma(S^3, \Lambda^1(2))$ and **twisted Euler**

$$0 \rightarrow \Lambda^1(2) \rightarrow \mathcal{E}_i(1) \xrightarrow{x^i} \mathcal{E}(2) \rightarrow 0$$

$\Rightarrow x^i \nabla_i f_j = f_j$ and $x^i f_i = 0$. Therefore,

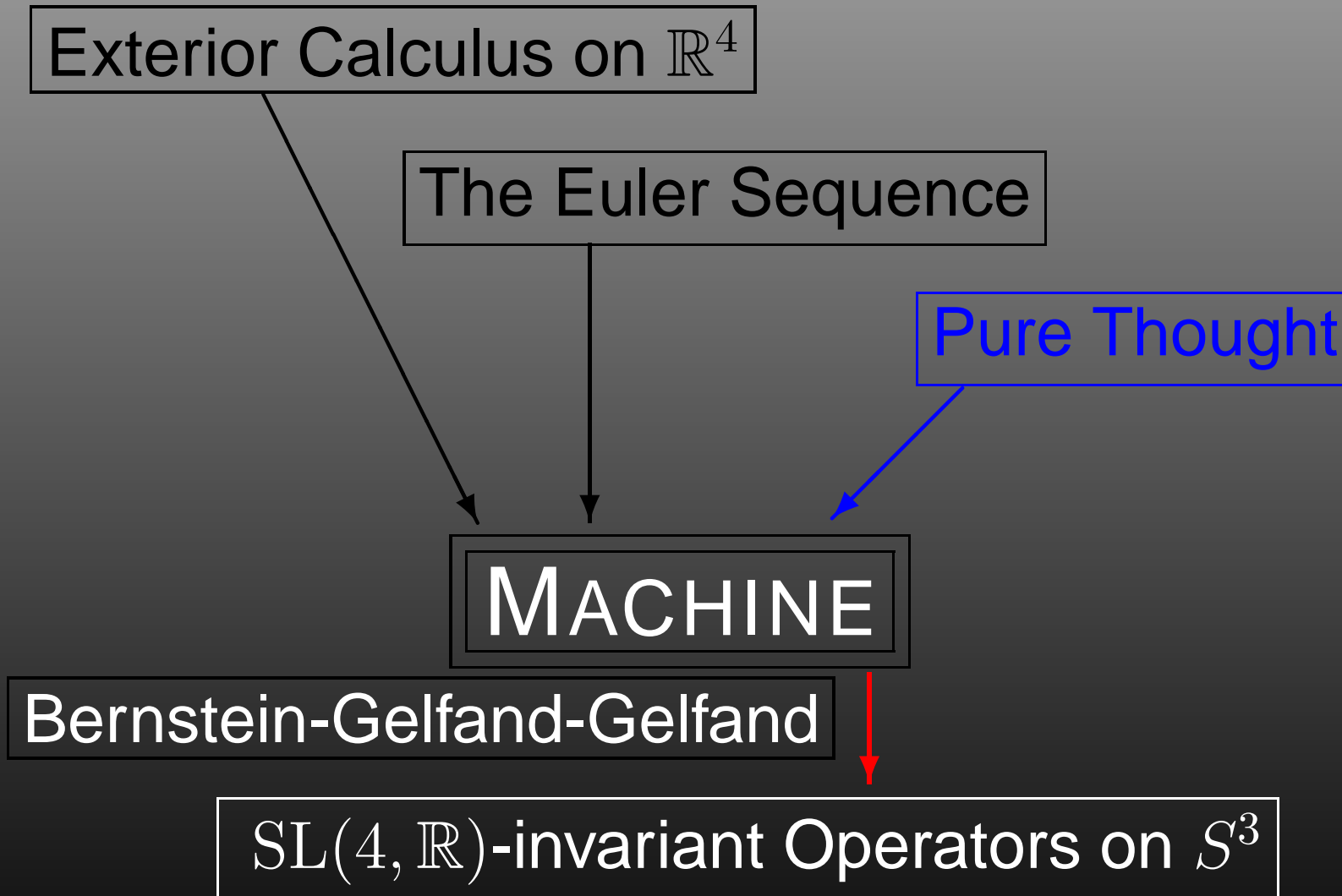
$$\begin{aligned} x^i \nabla_{(i} f_{j)} &= \frac{1}{2} (x^i \nabla_i f_j + \nabla_j (x^i f_i) - (\nabla_j x^i) f_i) \\ &= \frac{1}{2} (f_j + 0 - f_j) = 0. \end{aligned}$$

But $0 \rightarrow \odot^2 \Lambda^1(2) \rightarrow \mathcal{E}_{(ij)} \xrightarrow{x^i} \mathcal{E}_j(1) \rightarrow 0$ is **exact**

and so $\nabla : \Lambda^1(2) \rightarrow \odot^2 \Lambda^1(2)$ is invariant.



Moral



Classification!

Theorem We've found all $SL(4, \mathbb{R})$ -invariant linear differential operators on projective S^3 : if $D : E \rightarrow F$ is a non-constant invariant linear differential operator between irreducible homogeneous bundles, then

- D may be constructed by our machine
- there is an explicit list of such D .

Best proof \leftrightarrow representation theory of $\mathfrak{sl}(4, \mathbb{R})$.

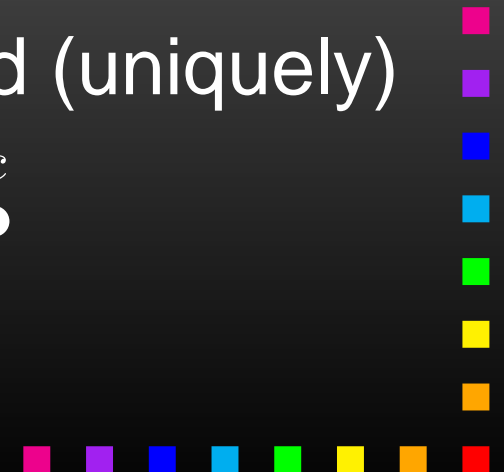


The list

For non-negative integers a, b, c , there are invariant operators

$$\begin{array}{ccc}
 a & b & c \\
 \times \text{---} \bullet \text{---} \bullet & \xrightarrow{\nabla(a+1)} & -a-2 \quad a+b+1 \quad c \\
 & & \times \text{---} \bullet \text{---} \bullet \\
 & & \xrightarrow{\nabla(b+1)} \quad -a-b-3 \quad a \quad b+c+1 \\
 & & \times \text{---} \bullet \text{---} \bullet \\
 & & \xrightarrow{\nabla(c+1)} \quad -a-b-c-4 \quad a \quad b \\
 & & \times \text{---} \bullet \text{---} \bullet .
 \end{array}$$

- All invariant operators are captured (uniquely)
- the kernel of the first one is $\begin{array}{ccc} a & b & c \\ \bullet & \text{---} & \bullet & \text{---} & \bullet \end{array}$
- it's locally exact! (BGG resolution)



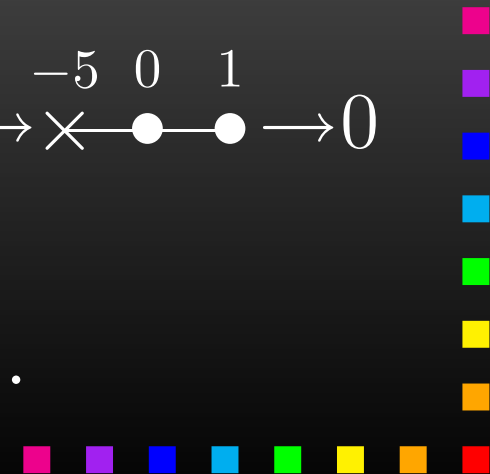
Examples

$a = 0, b = 0, c = 0 \rightsquigarrow$ de Rham

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & -3 & 0 & 1 & -4 & 0 & 0 & & \\
 0 \rightarrow & \bullet & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \rightarrow & 0 \\
 & \parallel & & & \parallel & & & \parallel & & & \parallel & & & \parallel & & & & \\
 0 \rightarrow & \mathbb{R} & \rightarrow & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \rightarrow & 0 & & & & & &
 \end{array}$$

$a = 0, b = 1, c = 0 \rightsquigarrow$ linear elasticity!

$$\begin{array}{ccccccc}
 & 0 & 1 & 0 & 0 & 1 & 0 & -2 & 2 & 0 & -4 & 0 & 2 & -5 & 0 & 1 & & \\
 0 \rightarrow & \bullet & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \times & \bullet & \bullet & \rightarrow & 0 \\
 & \parallel & & & \parallel & & & \parallel & & & & & & & & & & \\
 0 \rightarrow & \Lambda^2 \mathbb{R}^4 & \rightarrow & \Lambda^1(2) & \xrightarrow{\nabla} & \odot^2 \Lambda^1(2) & \xrightarrow{\nabla^{(2)}} & \dots & & & & & & & & & &
 \end{array}$$



What's happening on $\mathbb{R}^3 \hookrightarrow S^3$?

de Rham

$$f \mapsto \nabla_a f \quad f_a \mapsto \nabla_{[a} f_{b]} \quad f_{ab} \mapsto \nabla_{[a} f_{bc]}$$

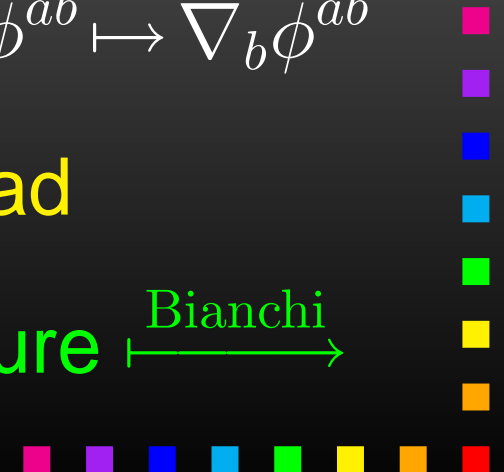
$$f \xrightarrow{\text{grad}} \nabla_a f \quad f_a \xrightarrow{\text{curl}} \epsilon^{abc} \nabla_b f_c \quad f^a \xrightarrow{\text{div}} \nabla_a f^a$$

linear elasticity = Riemannian deformation

$$\phi_a \mapsto \nabla_{(a} \phi_{b)} \quad \phi_{ab} \mapsto \epsilon^{ace} \epsilon^{bdf} \nabla_c \nabla_d \phi_{ef} \quad \phi^{ab} \mapsto \nabla_b \phi^{ab}$$

displacement \mapsto strain \mapsto stress \mapsto load

cöörd change $\xrightarrow{\text{Killing}}$ metric \mapsto curvature $\xrightarrow{\text{Bianchi}}$



Application to numerical analysis

Finite Element Exterior Calculus on \mathbb{R}^4 \equiv FEEC

The Euler Sequence

Sobolev Spaces

BGG MACHINE

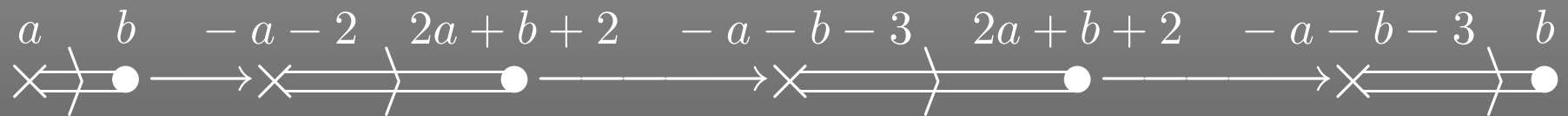
BGG FEEC

New Stable Finite Element Schemes for
Linear Elasticity (Arnold, Falk, & Winther)

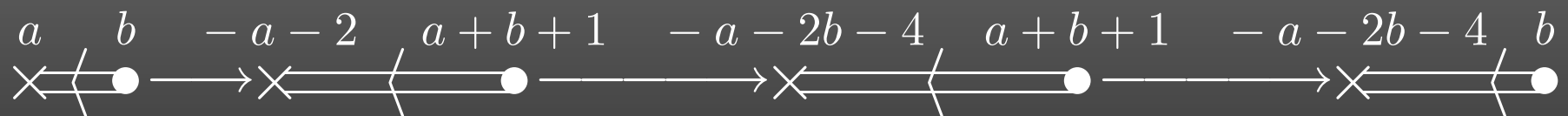


More BGG complexes on 3-sphere

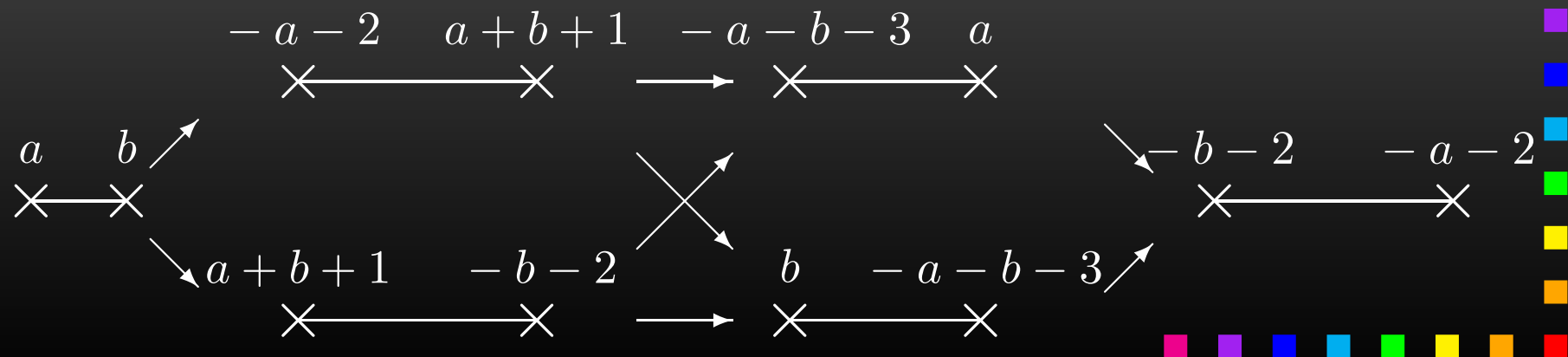
Conformal



Contact projective



CR



What's going on here?

- Representation theory of simple Lie groups.
- G is simple. P is parabolic.
- The (affine) action of the Weyl group of G .
- Hasse diagrams.
- Central character.
- The Jantzen-Zuckerman translation principle.
- Verma modules.
- Geometric interpretation.
- **Parabolic geometry**: Čap, Slovák, Souček,...



THANK YOU

