How to recognise the geodesics of a metric connection

Michael Eastwood

Australian National University

References

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Metric geodesics

metric $\longmapsto$ Levi-Civita connection $\longmapsto$ geodesics

$\downarrow$

unparameterised geodesics

What is lost? What do we obtain? Recall that

$$g_{ab} \mapsto \Gamma_{ab}{}^{d} = \tfrac{1}{2} g^{cd}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})$$

gives the Levi Civita connection

$$\nabla_a \phi_b = \partial_a \phi_b - \Gamma_{ab}{}^{c} \phi_c.$$

Projective equivalence

Suppose ∇_a is a torsion-free connection. Define $\hat{\nabla}_a$ by

$$\boxed{\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a.}$$

- $\hat{\nabla}_a$ is torsion-free.
- $\hat{\nabla}_a$ has the same unparameterised geodesics as ∇_a.
- Conversely, these two properties force $\boxed{}$.

Definition: ∇_a and $\hat{\nabla}_a$ are projectively equivalent.

Our questions are now operational:

$$\sigma : \{\text{metrics}\} \stackrel{?}{\longmapsto} \{\text{projective structures}\}.$$

Example: σ is not injective

$\sigma(g_{ab}) = \sigma(\text{constant} \times g_{ab})$ but also

$$\frac{(1-y^2)\,dx^2 + xy\,dx\,dy + (1-x^2)\,dy^2}{(1-x^2-y^2)^2} \qquad \text{Beltrami}$$

on $\{x^2+y^2<1\}$ has the same unparameterised geodesics as

$$\frac{(1+y^2)\,dx^2 - xy\,dx\,dy + (1+x^2)\,dy^2}{(1+x^2+y^2)^2} \qquad \text{Thales}$$

or, indeed, as the flat metrics

$$dx^2 + dy^2 \quad \text{or} \quad dx^2 - dy^2.$$

Example: σ is not surjective

Writing $\nabla_a \phi_b = \partial_a \phi_b - \Gamma_{ab}{}^c \phi_c$,

$$\Gamma_{ab}{}^1 = \frac{1}{1+x^2} \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \qquad \Gamma_{ab}{}^2 = \frac{1}{1+x^2} \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$

is a metric connection but

$$\Gamma_{ab}{}^1 = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \qquad \Gamma_{ab}{}^2 = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

is not projectively equivalent to a metric connection.

Why Not?

Another example

The connection

$$\Gamma_{ab}{}^{1} = \begin{bmatrix} \dfrac{xy + 2e^y + 2ye^y + 2x^2ye^y}{1 + y + x^2y} & x + y^2 \\ x + y^2 & -\dfrac{x}{2(1 + y + x^2y)} \end{bmatrix}$$

$$\Gamma_{ab}{}^{2} = \begin{bmatrix} \dfrac{1}{1 + y + x^2y} & e^y \\ e^y & \dfrac{1 + x^2 + 4x + 4xy + 4x^3y + 4y^2 + 4y^3 + 4y^3x^2}{2(1 + y + x^2y)} \end{bmatrix}$$

is projectively equivalent to a metric connection!

Naïve dimension count

On a surface

$$\begin{array}{rcccccccccccc} \text{Jets} & = & \Lambda^0 & + & \Lambda^1 & + & \bigodot^2\Lambda^1 & + & \bigodot^3\Lambda^1 & + & \bigodot^4\Lambda^1 & + & \cdots \\ \text{rank} & = & 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & \cdots \end{array}$$

Therefore

k	1	2	3	4	5	6	7
rankJ^{k+1}	6	10	15	21	28	36	45
rankJ^{k+1}(metrics)	18	30	45	63	84	108	135
ditto up to scale	17	29	44	62	83	107	134
rankJ^k(proj. struc.)	12	24	40	60	84	112	144
deficit	–	–	–	–	1	5	10

Obstructions

We expect that

$$J^{k+1}(\text{metrics up to scale}) \longrightarrow J^k(\text{projective structures})$$

is not surjective for $k = 5$. Therefore, we expect

$$\boxed{\mathcal{D}(\Gamma) = \text{polynomial in } \Gamma, \nabla\Gamma, \cdots, \nabla^{(5)}\Gamma}$$

such that Γ metrisable $\Rightarrow \mathcal{D}(\Gamma) = 0$. For $k = 6$, expect

$$5 = \#\{\mathcal{D}(\Gamma), \nabla_1\mathcal{D}(\Gamma), \nabla_2\mathcal{D}(\Gamma)\} + \boxed{\text{two more invariants}}$$

and then $10 < 6 + 6 \Rightarrow$ generically this is enough (real-analytic . . .).

Special connections

Ricci curvature

$$(\nabla_a\nabla_b - \nabla_b\nabla_a) = -R_{ab}X^b$$

is not necessarily symmetric. However,

$$\hat{\nabla}_a\phi_b = \nabla_a\phi_b - \Upsilon_a\phi_b - \Upsilon_b\phi_a$$

implies

$$\hat{R}_{[ab]} = R_{[ab]} - (n+1)\nabla_{[a}\Upsilon_{b]}.$$

Bianchi $\Rightarrow \nabla_{[a}R_{bc]} = 0$. Thus, restrict to $R_{ab} = R_{(ab)}$.

Definition Connections with symmetric Ricci $\equiv$ special.

An overdetermined system

Theorem (Liouville, . . . , Sinjukov, Mikeš, . . .)
A special connection ∇_a is projectively equivalent to a metric connection if and only if the system of PDEs

$$\text{trace-free part of } (\nabla_a\sigma^{bc}) = 0$$

$$\nabla_a\sigma^{bc} - \frac{1}{n+1}\delta_a{}^b\nabla_d\sigma^{cd} - \frac{1}{n+1}\delta_a{}^c\nabla_d\sigma^{bd}$$

for σ^{ab} symmetric has a non-degenerate solution.
Proof Check that $g^{ab} = \det(\sigma)\sigma^{ab}$ works, where

$$\det(\sigma) = \epsilon_{a\cdots b}\epsilon_{c\cdots d}\sigma^{ac}\cdots\sigma^{bd}$$

and $\nabla_a\epsilon_{c\cdots d} = 0$ (OK for special connections). $\square$

Prolongation

Rewrite as

$$\nabla_a \sigma^{bc} = \delta_a{}^b \mu^c + \delta_a{}^c \mu^b.$$

Flat case as warm-up:–

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma^{bc} = n\nabla_a \mu^c - \delta_a{}^c \nabla_b \mu^b$$

whence

$$\nabla_a \mu^b = \delta_a{}^b \rho.$$

But

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a)\mu^b = (n-1)\nabla_a \rho$$

whence

$$\nabla_a \rho = 0.$$

Tractor connection

Curved case:–

$$\begin{aligned} \nabla_a \sigma^{bc} &= \delta_a{}^b \mu^c + \delta_a{}^c \mu^b \\ \nabla_a \mu^b &= \delta_a{}^b \rho - \mathrm{P}_{ac} \sigma^{bc} + \tfrac{1}{n} W_{ac}{}^b{}_d \sigma^{cd} \\ \nabla_a \rho &= -2\mathrm{P}_{ab} \mu^b + \tfrac{4}{n} Y_{abc} \sigma^{bc} \end{aligned}$$

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} \qquad Y_{abc} = \nabla_{[a} \mathrm{P}_{b]c}.$$

Reorganize as tractors:–

$$\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} \xmapsto{\nabla_a} \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a{}^b \mu^c - \delta_a{}^c \mu^b \\ \nabla_a \mu^b - \delta_a{}^b \rho + \mathrm{P}_{ac} \sigma^{bc} \boxed{- \tfrac{1}{n} W_{ac}{}^b{}_d \sigma^{cd}} \\ \nabla_a \rho + 2\mathrm{P}_{ab} \mu^b \boxed{- \tfrac{4}{n} Y_{abc} \sigma^{bc}} \end{pmatrix}$$

cf. Cartan connection.

Surfaces

$W_{ab}{}^c{}_d = 0$ and Cotton-York Y_{abc} is projectively invariant.

$$\Sigma^\alpha \equiv \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} \xrightarrow{\nabla_a} \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a{}^b \mu^c - \delta_a{}^c \mu^b \\ \nabla_a \mu^b - \delta_a{}^b \rho + \mathrm{P}_{ac} \sigma^{bc} \\ \nabla_a \rho + 2\mathrm{P}_{ab} \mu^b - 2Y_{abc} \sigma^{bc} \end{pmatrix}$$

Compute curvature (fix a volume form ϵ^{ab}):–

$$\epsilon^{ab} \nabla_a \nabla_b \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5Y_a \mu^a + Z_{ab} \sigma^{ab} \end{pmatrix}$$

where $Y_a = \epsilon^{bc} Y_{bca}$ and $Z_{ab} = \nabla_{(a} Y_{b)}$.

Consequences of metrisability

If Γ is metrisable, then $\nabla_a \Sigma^\alpha = 0$ for some $\Sigma^\alpha \not\equiv 0$. Now

$$\epsilon^{ab}\nabla_a\nabla_b\Sigma^\alpha = 0 \Rightarrow \Xi_\alpha\Sigma^\alpha = 0 \ \text{ where } \ \Xi_\alpha \equiv \begin{pmatrix} 0 \\ 5Y_a \\ Z_{ab} \end{pmatrix}.$$

Differentiate once more:–

$$0 = \nabla_a(\Xi_\gamma\Sigma^\gamma) = (\nabla_a\Xi_\gamma)\Sigma^\gamma \quad \text{(dual connection)}$$

$$\nabla_a\Sigma_\gamma = \begin{pmatrix} 5Y_a \\ 5\nabla_a Y_c + 2Z_{ac} \\ \nabla_a Z_{cd} - 5\mathrm{P}_{a(c}Y_{d)} \end{pmatrix}.$$

First obstruction

Differentiate again:–

$$0 = (\nabla_{(a}\nabla_{b)}\Xi_\gamma)\Sigma^\gamma.$$

The 6×6 matrix

$$\left(\Xi_\gamma, \nabla_a\Xi_\gamma, \nabla_{(a}\nabla_{b)}\Xi_\gamma\right)$$

is singular. The 5^{th} order expression

$$\boxed{\mathcal{D}(\Gamma) \equiv \det\left(\Xi_\gamma, \nabla_a\Xi_\gamma, \nabla_{(a}\nabla_{b)}\Xi_\gamma\right)}$$

is a projectively invariant obstruction to metrisability.

Example

Can compute with MAPLE.

$$\Gamma_{ab}{}^{1} = \begin{bmatrix} x & x+y \\ x+y & y \end{bmatrix} \qquad \Gamma_{ab}{}^{2} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

is not metrisable because is has non-vanishing primary obstruction:–

$$
\begin{aligned}
\mathcal{D}(\Gamma) = -\frac{32}{19683} e^{-6y^2-3x^2-6xy} \Big(& 54648740x^2-307413592xy-71109308y^2 \\
& +959364y^{12}+784892430x^{12}-87474336x^3y+1133803492x^2y^2+1003230736xy^3 \\
& +880266528x^{10}y^6-42320448x^5y^9+217229904x^5y^7-412345728x^{13}y^3-244856520x^{14} \\
& -64595700y^9x+86106087y^8x^2+141717600x^{16}+7004232x^2y^{12}-933120xy^{13} \\
& -1465143957x^{10}+52488y^{14}-2521404720x^7y^3+69984x^2y^{14}+1178017344x^{11}y^3 \\
& +154546056x^7y^5-240504696x^5y^5-885577536x^{13}y-166793688x^6y^8+362722626x^4y^6 \\
& -3682383255x^8y^2-1132982640x^9y^3-960771456x^9y^5+637151832x^{10}y^4-111087468x^6y^6 \\
& +26135136x^6y^{10}+2531159442x^{10}y^2-3419365428x^9y-32904576x^5y^{11}+9803808x^4y^{12} \\
& +32691870x^4y^8+418345989x^6+1115929062x^8-3565116y^8-108590703y^6-80492982y^7x \\
& -781618710y^5x-172461312x^9y^7+699140160x^{15}y+63073512y^{10}x^4-306669024x^8y^8 \\
& -12083904y^{11}x+135746496x^7y^9-1059525360x^8y^4+3132002700x^{11}y-1321920x^3y^{13} \\
& +918959904x^{14}y^2-1020384738x^6y^4+130963392x^{11}y^5-469505160x^{12}y^2+109834704x^3y^7 \\
& +151175418x^4+213452946y^4+455151744x^7y^7-121150260x^3y^9-1323467424x^{12}y^4 \\
& +4388454282x^6y^2+791844546x^3y^5-2367092292x^3y^3-1892941254x^5y+2905986870x^7y \\
& +2283906366x^5y^3+1959196446x^4y^4-1297857849x^4y^2-1956344445x^2y^4+285904686y^6x^2 \\
& -111011688x^8y^6-28451520y^{11}x^3+7720677y^{10}+58471578y^{10}x^2+73919880 \Big)
\end{aligned}
$$

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