The X-ray transform: part I

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[based on work with Toby Bailey
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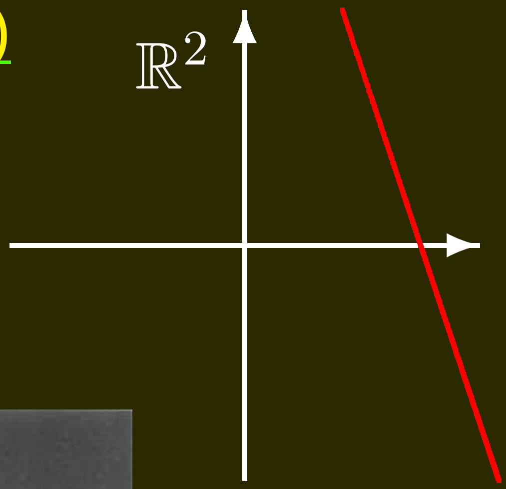
Radon transform

Johann Radon (1917)



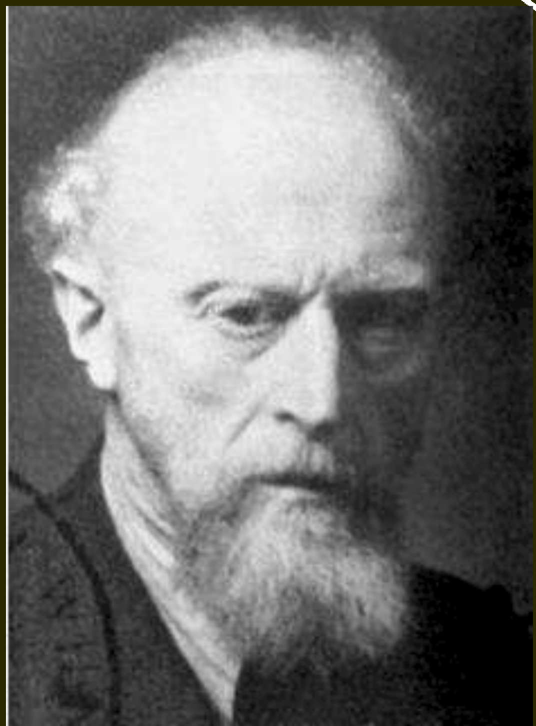
$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E}) \qquad \phi(\gamma) = \int_\gamma f$$

Funk transform

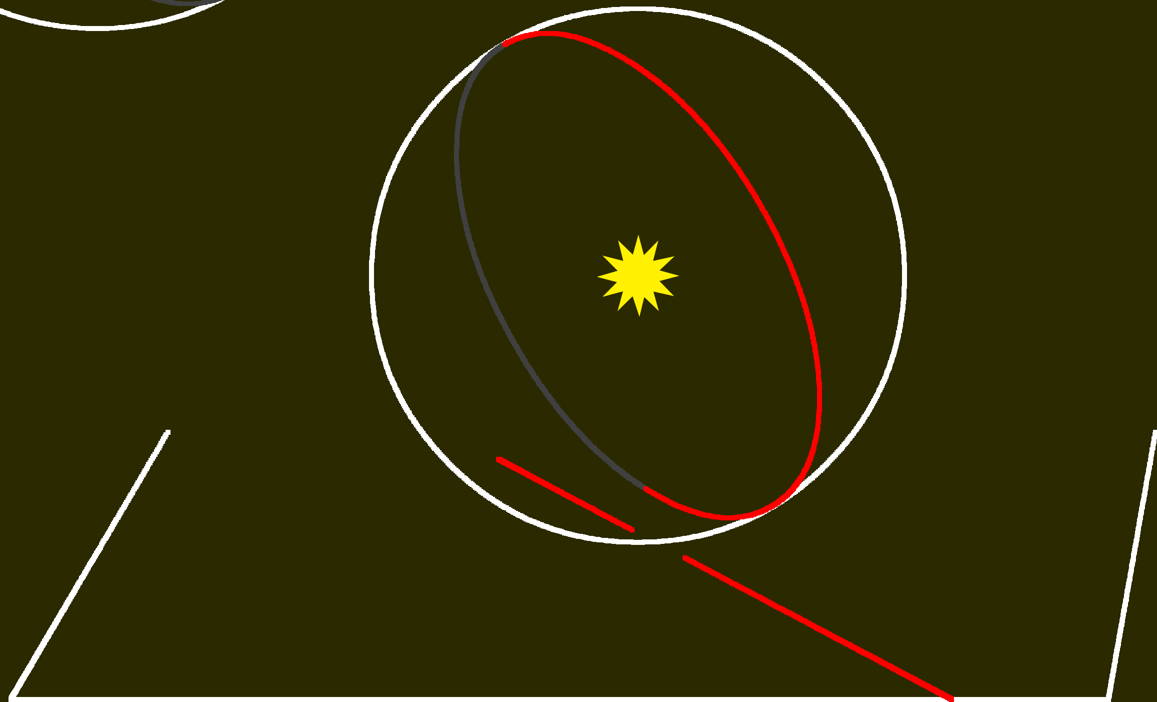
Paul Funk (1913)

$$\phi(\gamma) = \oint_\gamma f$$

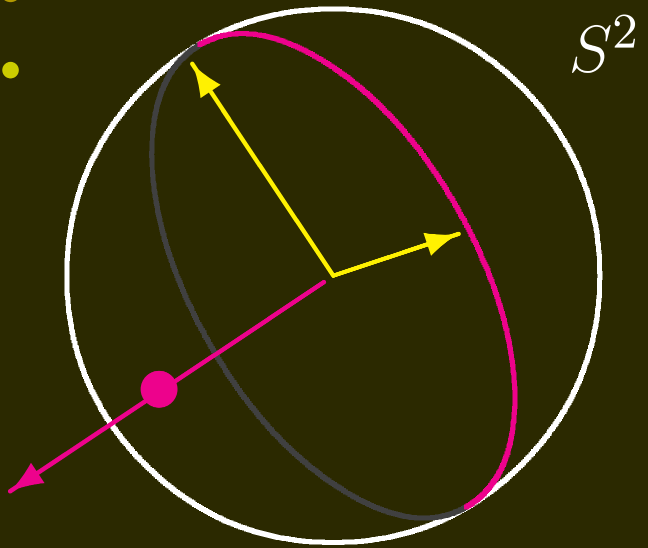
$$f \in \Gamma(S^2, \mathcal{E})$$



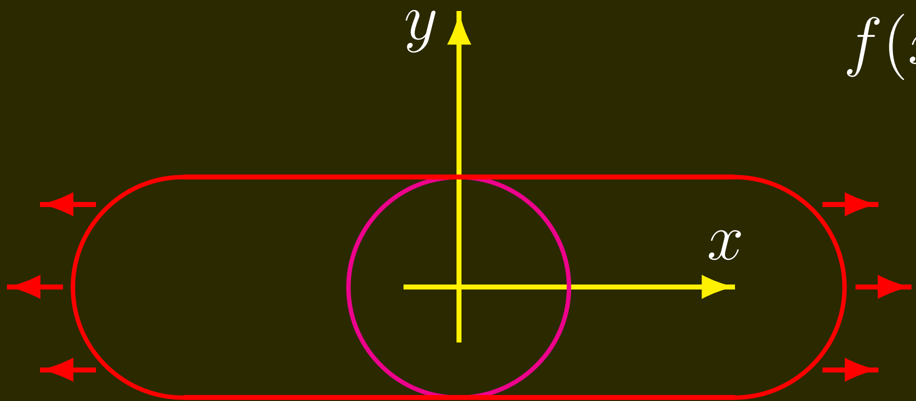
Radon=Funk!



Radon=Funk



$$\begin{array}{cccc}
\mathcal{F}:\Gamma(S^2,\mathcal{E}) & \longrightarrow & \Gamma(\mathbb{RP}_2,\mathcal{E}) & \text{good} \\
\text{even}\big\downarrow\text{part} & & \big\| & \\
\mathcal{F}:\Gamma(\mathbb{RP}_2,\mathcal{E}) & \xrightarrow[\text{Funk}]{\simeq} & \Gamma(\mathbb{RP}_2,\mathcal{E}) & \text{better} \\
\wr\| & & \wr\| & \\
\mathcal{F}:\Gamma(\mathbb{RP}_2,\mathcal{E}(-2)) & \to & \Gamma(\mathbb{RP}_2^*,\widetilde{\mathcal{E}}[-1]) & \text{best!}
\end{array}$$



$f(x,y)$ homogeneous of degree -2

$$f(\lambda x,\lambda y)=\lambda^{-2}f(x,y)\ \ \forall\,\lambda\neq 0$$

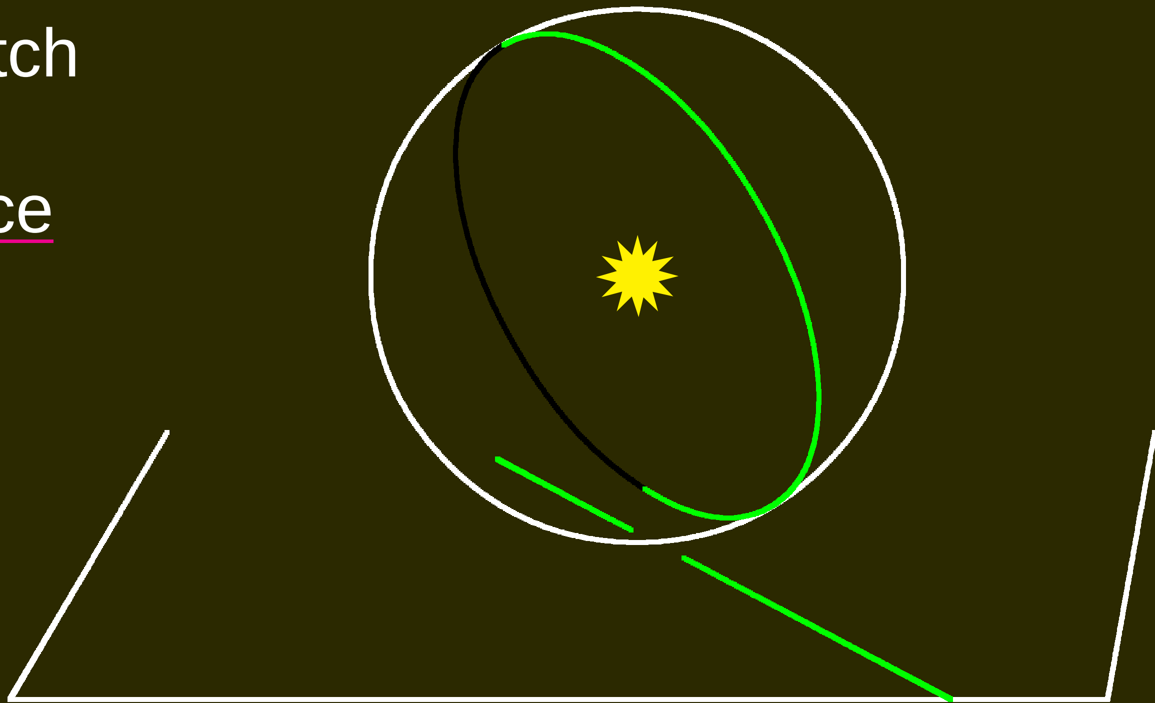
$f(x,y)(x\,dy-y\,dx)$ is closed (Euler)

Projective differential geometry

Def$^{\rm n}$ $\hat{\nabla}_a \sim \nabla_a \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) the round sphere is projectively flat

Affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a projective equivalence



$$\frac{(1+y^2)\,dx^2 - xy\,dx\,dy + (1+x^2)\,dy^2}{(1+x^2+y^2)^2}$$

Symmetry groups

$\boxed{\mathbb{R}^2}$ Euclidean symmetries $\vec{x} \mapsto A\vec{x} + \vec{b}$ for $A \in \mathrm{SO}(2)$ NB

Affine symmetries $\vec{x} \mapsto A\vec{x} + \vec{b}$ for $A \in \mathrm{SL}(2, \mathbb{R})$

$\rightsquigarrow$ Fourier transform ($\checkmark$ Radon)

dimensions agree!

$\boxed{S^2}$ Isometries $\mathbb{R}^3 \ni \vec{v} \mapsto M\vec{v}$ for $M \in \mathrm{SO}(3)$ NB

$\rightsquigarrow$ Expand in spherical harmonics ($\checkmark$ Funk)

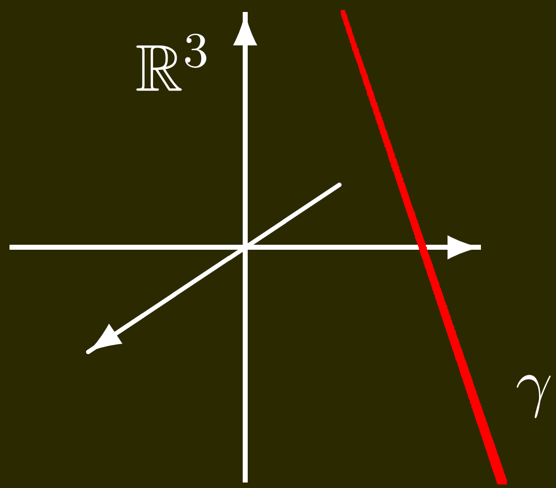
$\boxed{\mathbb{RP}_2}$ Projective symmetries $\mathbb{R}^3 \ni \vec{v} \mapsto M\vec{v}$ for $M \in \mathrm{SL}(3, \mathbb{R})$

$\rightsquigarrow$ to be explained . . . ($\checkmark$ bigger is better)

$$\mathrm{SO}(3) \subset \mathrm{SL}(3, \mathbb{R}) \supset \left\{ \begin{bmatrix} A & \vec{b} \\ \vec{0}^{\,t} & 1 \end{bmatrix} \right\} \qquad \begin{bmatrix} A & \vec{b} \\ \vec{0}^{\,t} & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = \begin{bmatrix} A\vec{x} + \vec{b} \\ 1 \end{bmatrix}$$

X-ray transform

Fritz John (1938)



$$f \in \Gamma_*(\mathbb{R}^3, \mathcal{E}) \qquad \phi(\gamma) = \int_\gamma f$$

$$\phi(w,x,y,z) = \int_{-\infty}^{\infty} f(w+xt, y+zt, t)\, dt$$

$$\Box\phi \equiv \frac{\partial^2\phi}{\partial w\,\partial z} - \frac{\partial^2\phi}{\partial x\,\partial y} = 0$$

Ultrahyperbolic wave equation

Theorem J

X-ray transform (compactify like Funk)

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1])$$

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^4) & \\ \swarrow & & \searrow \\ \mathbb{RP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

correspondence

double fibration

Th$^{\mathrm{m}}$ $\exists!$ $\mathrm{SL}(4,\mathbb{R})$-invariant linear differential operator

$$\Box : \widetilde{\mathcal{E}}[-1] \to \widetilde{\mathcal{E}}[-3] \quad \text{on } \mathrm{Gr}_2(\mathbb{R}^4).$$

Th$^{\mathrm{m}}$ J There is an exact sequence

$$0 \to \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\mathcal{X}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1]) \xrightarrow{\Box} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3]).$$

Penrose transform

Harry Bateman (1904)

$$\phi(\underbrace{w,x,y,z}_{\in\mathbb{R}^4}) = \oint_\gamma f\left((w+ix)+(iy+z)\zeta,(iy-z)+(w-ix)\zeta,\zeta\right)d\zeta$$

☞ holomorphic function of 3-variables

[cf. John $\int_{-\infty}^{\infty} f(w+xt,y+zt,t)\,dt$]

$$\Delta\phi = 0$$

Laplacian

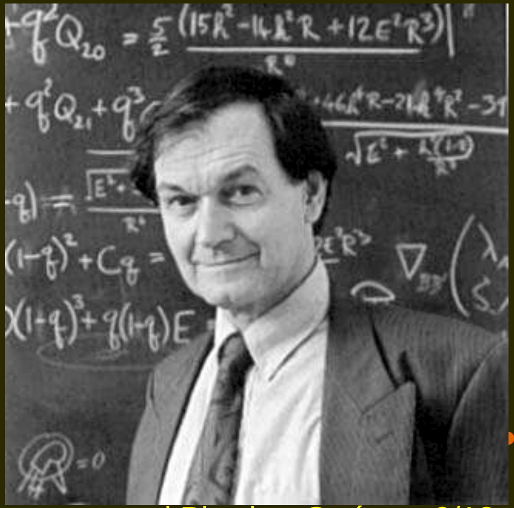
Roger Penrose (1970s)

$$\phi(\underbrace{w,x,y,z}_{\in\mathbb{R}^4}) = \oint_\gamma f\left((w+x)+(y+iz)\zeta,(y-iz)+(w-x)\zeta,\zeta\right)d\zeta$$

☞ holomorphic function of 3-variables

$$\frac{\partial^2\phi}{\partial w^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Wave equation



Theorem EPW

$$\mathbb{F}_{1,2}(\mathbb{C}^4)$$

$$\swarrow \qquad \searrow$$

$$\mathbb{CP}_3 \qquad\qquad \mathrm{Gr}_2(\mathbb{C}^4) \equiv \mathbb{M}$$

$\boxed{\mathrm{Th}^{\mathrm{m}}}$ $\exists!$ $\mathrm{SL}(4,\mathbb{C})$-invariant linear differential operator

$$\Box : \mathcal{O}[-1] \to \mathcal{O}[-3] \quad \text{on } \mathbb{M}.$$

$\mathrm{SU}(2,2)$-orbits

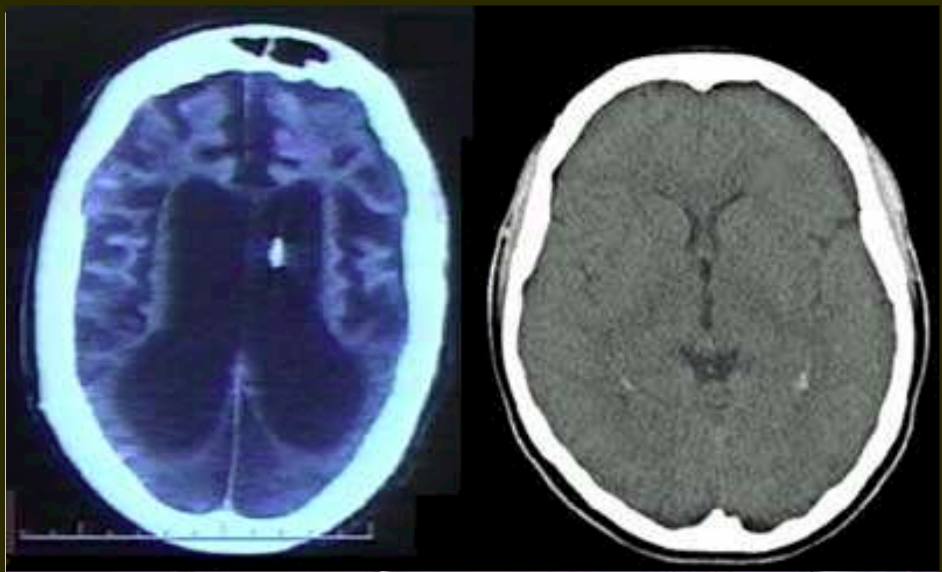
$$\mathbb{CP}_3 = \mathbb{CP}_3^+ \cup \mathbb{CP}_3^- \cup \mathbb{CP}_3^0$$

$$\mathbb{M} = \mathbb{M}^{++} \cup \mathbb{M}^{+-} \cup \mathbb{M}^{--} \cup \mathbb{M}^{+0} \cup \mathbb{M}^{0-} \cup \mathbb{M}^{00}$$

$\boxed{\mathrm{Th}^{\mathrm{m}}\ \mathrm{EPW}}$ There is an exact sequence

$$0 \longrightarrow H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \xrightarrow{\mathcal{P}} \Gamma(\mathbb{M}^{++}, \mathcal{O}[-1]) \xrightarrow{\Box} \Gamma(\mathbb{M}^{++}, \mathcal{O}[-3]).$$

X-rays in action!



END OF PART ONE

THANK YOU