The X-ray transform: part II

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The simplest X-ray transform is a version of the Radon transform in three dimensions. One starts with suitably decaying function of three variables and integrates it over the lines in Euclidean three-space obtaining a function on the four-dimensional space of lines. This transform is often named after John who identified its range in 1938. There are many variations on this theme! There is a compactified version, due to Funk in 1913. There is a complex version, due to Bateman in 1904. Nowadays, there are all sorts of X-ray transforms and the purpose of these lectures will be to describe the links between them and to use representation theory and differential geometry to establish their range and kernel in various cases.

References

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Recall Lecture I

X-ray transform

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^4) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{RP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

correspondence

$$\mathrm{Gr}_2(\mathbb{R}^4) \ni x \mapsto \alpha(\beta^{-1}(x))$$

geodesic on $\mathbb{RP}_3$

$\boxed{\mathrm{Th}^{\mathrm{m}}\ \mathrm{J}}$ There is an exact sequence

$$0 \to \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\mathcal{X}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1]) \xrightarrow{\Box} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3]).$$

Penrose transform

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{C}^4) \end{array}$$

NB

$$\mathbb{RP}_3 \hookrightarrow \mathbb{CP}_3$$
$$\mathcal{O}(-2)|_{\mathbb{RP}_3} = \mathcal{E}(-2)$$

$$H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \xrightarrow{\mathcal{P}} \cdots$$

Machinery for the Penrose transform

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & \\ {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\ \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{C}^4) \end{array}$$

V = holomorphic vector bundle on $\mathbb{CP}_3$

$$0 \to \mu^{-1}\mathcal{O}(V) \to \Omega^\bullet_\mu \otimes \mu^* V$$

$$\boxed{E_1^{p,q} = \nu_*^q(\Omega^p \otimes \mu^* V) \Longrightarrow H^{p+q}(\mathcal{O}(V))}$$

Easy to use (when V is irreducible homogeneous)

$$\Omega^\bullet_\mu \qquad \mathcal{O} \xrightarrow{d_\mu} \mathcal{O}_A(1)[-1] \xrightarrow{d_\mu} \mathcal{O}(2)[-3] \qquad \text{on } \mathbb{F}_{1,2}(\mathbb{C}^4)$$

$$\mathcal{O}(-3) \quad \rightsquigarrow \quad \begin{array}{ccccc} \mathcal{O}(-3) & \xrightarrow{d_\mu} & \mathcal{O}_A(-2)[-1] & \xrightarrow{d_\mu} & \mathcal{O}(-1)[-3] \\ \downarrow \nu_*^1 & & \downarrow \nu_*^1 & & \\ \mathcal{O}_{A'}[-1] & \xrightarrow{\not{D}} & \mathcal{O}_A[-2] & & \end{array}$$

Dirac operator!

on $\mathbb{M} = \mathrm{Gr}_2(\mathbb{C}^4)$

Examples of the Penrose transform

$$\underline{\mathcal{O}(-2)} \quad \rightsquigarrow \quad \mathcal{O}(-2) \xrightarrow{d_\mu} \mathcal{O}_A(-1)[-1] \xrightarrow{d_\mu} \mathcal{O}[-3]$$

$$\downarrow \nu^1_* \qquad\qquad\qquad\qquad \downarrow \nu^0_*$$

$$\mathcal{O}[-1] \xrightarrow{\qquad \Box \qquad} \mathcal{O}[-3]$$

Th$^{\mathrm{m}}$ EPW

$$\underline{\mathcal{O}} \quad \rightsquigarrow \quad \Omega^0_\mu \xrightarrow{d_\mu} \Omega^1_\mu \xrightarrow{d_\mu} \Omega^2_\mu$$

$$\downarrow \nu^0_* \qquad \downarrow \nu^0_* \qquad \downarrow \nu^0_*$$

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_+} \Omega^2_+$$

$$H^1(\mathbb{CP}^+, \mathcal{O}) \ \downarrow\wr \ \Gamma(\mathbb{M}^{++}, \mathrm{Max}_-)$$

$$\underline{\Omega^1} \quad \rightsquigarrow \quad \overset{-2}{\times}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{0}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \overset{1}{\times}\!-\!\overset{-4}{\times}\!-\!\overset{1}{\bullet}$$

BGG

$$\downarrow \nu^1_* \qquad \downarrow \nu^0_* \qquad \downarrow \nu^0_*$$

$$\Omega^0$$

$$\Omega^2_- \xrightarrow{d} \Omega^3$$

exact sequence

$$0 \to H^1(\mathbb{CP}^+_3, \mathcal{O}) \xrightarrow{d} H^1(\mathbb{CP}_3, \Omega^1) \to \Gamma(\mathbb{M}^{++}, \Omega^0) \xrightarrow{\Box^2} \Gamma(\mathbb{M}^{++}, \Omega^4)$$

Geometry of the Penrose/X-ray transform

$$\begin{array}{ccccc} & & \mathbb{F}_{1,2}(\mathbb{C}^4) & & \\ & \swarrow & & \searrow & \\ \mathbb{CP}_3 & & & & \mathrm{Gr}_2(\mathbb{C}^4) \end{array}$$

$\mathrm{SL}(2,\mathbb{H})$-orbits

$$\begin{array}{ccccc} & & \nu^{-1}(S^4) & & \\ & \swarrow\!\!\swarrow & & \searrow & \\ \mathbb{CP}_3 & & \xrightarrow{\tau} & & S^4 \end{array}$$

$\mathrm{SL}(4,\mathbb{R})$-orbits

another parabolic geometry!

$$\begin{array}{ccccc} & & \mathbb{F}_{1,2}(\mathbb{R}^4) & & \\ & \swarrow & & \searrow & \\ \mathbb{RP}_3 & & & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

projective geometry

conformal geometry

$\mathrm{SL}(4,\mathbb{R}) = \mathrm{Spin}(3,3)$

Plücker: $\mathrm{Gr}_2(\mathbb{R}^4) \hookrightarrow \mathbb{RP}_5$

cf. Charles Frances' talks

Machinery for the X-ray transform

Complex analysis comes into play in two ways:

- constructing a spectral sequence,
- computing with the spectral sequence.

More generally:

$$\mathbb{F}_{1,2}(\mathbb{C}^{n+1}) \quad \xrightarrow{\mu} \mathbb{CP}_n, \quad \xrightarrow{\nu} \mathrm{Gr}_2(\mathbb{C}^{n+1})$$

$$\supset$$

$$\nu^{-1}(\mathrm{Gr}_2(\mathbb{R}^{n+1})) \equiv F \quad \xrightarrow{\eta} \mathbb{CP}_n, \quad \xrightarrow{\tau} \mathrm{Gr}_2(\mathbb{R}^{n+1})$$

$$\dim_{\mathbb{R}} = 2n$$

$$\cup$$

$$\mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \quad \to \mathbb{RP}_n, \quad \to \mathrm{Gr}_2(\mathbb{R}^{n+1})$$

a correspondence

but not a double fibration

Real blow up

$$F = \left\{ (L,P) \text{ s.t. } \begin{array}{l} L \subset \mathbb{C}^{n+1} \text{ is a complex line} \\ P \subset \mathbb{R}^{n+1} \text{ is a real plane} \\ \Re(L) \subseteq P \text{ (generic equality)} \end{array} \right\}$$

$$\downarrow \eta \qquad\qquad\qquad \downarrow$$

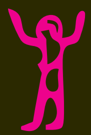
$$\mathbb{CP}_n = \{ L \text{ s.t. } L \subset \mathbb{C}^{n+1} \text{ is a complex line} \}$$

$$\begin{array}{ccc} F & \supset & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \\ \downarrow \eta & & \downarrow \\ \mathbb{CP}_n & \supset & \mathbb{RP}_n \end{array}$$

Real blow up of $\mathbb{CP}_n$ along $\mathbb{RP}_n$!

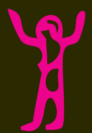
Involutive structure

- complex manifold Ω
  - $J : T\Omega \to T\Omega$ s.t. $J^2 = -\mathrm{Id} \ldots$



$\left\{ \begin{array}{l} \bullet\ T^{0,1} \subset \mathbb{C}T\Omega \text{ s.t. } [T^{0,1}, T^{0,1}] \subseteq T^{0,1} \\ \bullet\ \Lambda^{0,0} \xrightarrow{\bar\partial} \Lambda^{0,1} \xrightarrow{\bar\partial} \Lambda^{0,2} \longrightarrow \cdots \text{ s.t. } \bar\partial^2 = 0 \end{array} \right.$

- totally real submanifold $M \hookrightarrow \Omega$
  - $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \Omega$ and $TM \cap JTM = 0$



$\equiv$ involutive structure $\xrightarrow{\checkmark}$

$$\begin{array}{ccc} \widetilde{\Omega} & \supset & \Sigma \\ \downarrow \eta & & \downarrow \\ \Omega & \supset & M \end{array}$$

real blow-up

Involutive cohomology $H^r_{\bar\partial}(\widetilde{\Omega})$ (cf. Dolbeault, $\bar\partial_b$, ...)

The X-ray machine

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ \mathbb{RP}_n \subset \mathbb{CP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \end{array}$$

$V =$ holomorphic vector bundle on $\mathbb{CP}_n$

Pull-back to F

$$0 \to \Gamma(\mathbb{CP}_n, \mathcal{O}(V)) \to \Gamma(\mathbb{RP}_n, \mathcal{E}(V)) \to H^1_\partial(F, \widetilde{V}) \to H^1(\mathbb{CP}_n, \mathcal{O}(V)) \to 0$$

Example $\Gamma(\mathbb{RP}_n, \mathcal{E}(-2)) \xrightarrow{\simeq} H^1_{\bar\partial}(F, \text{Œ}(\widetilde{-2}))$

Push-down to $\mathrm{Gr}_2(\mathbb{R}^{n+1})$

$$E_1^{p,q} = \Gamma(\mathrm{Gr}_2(\mathbb{R}^{n+1}), \tau_*^q \text{Œ}_\eta^p(\widetilde{V})) \Longrightarrow H_{\bar\partial}(F, \widetilde{V})$$

Just like the Penrose transform!!

Examples of the X-ray transform

$\mathcal{E}(-2) \quad \rightsquigarrow \quad \mathrm{Th}^{\mathrm{m}}\ \mathrm{J}$

$$0 \to \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\mathcal{X}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1]) \xrightarrow{\Box} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3])$$

$\mathcal{E}(-3)$

$$0 \to \Gamma(\mathbb{RP}_3, \mathcal{E}(-3)) \xrightarrow{\mathcal{X}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}_{A'}[-1]) \xrightarrow{\not{D}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}_{A}[-2])$$

Λ^1

$$0 \to \mathbb{R} \to \Gamma(\mathbb{RP}_3, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{RP}_3, \Lambda^1) \xrightarrow{\mathcal{X}} \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\Lambda}^0)$$

$$\Box^2 \downarrow$$

$$\Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\Lambda}^4)$$

Unique conformally covariant operator: $\widetilde{\Lambda}^0 \to \widetilde{\Lambda}^4$
cf. Jean-Louis Clerc's talks

X-ray kernels (Bailey-E 1997)

Theorem The X-ray transform is injective on $\Gamma(\mathbb{RP}_n, \Lambda^0(-2))$ and on various other tensor fields:

$$0 \to \mathbb{R} \to \Gamma(\mathbb{RP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{RP}_n, \Lambda^1) \xrightarrow{\mathcal{X}}$$

R. Michel (1978)

$$0 \to \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \to \Gamma(\mathbb{RP}_n, \Lambda^1(2)) \xrightarrow{\nabla} \Gamma(\mathbb{RP}_n, \textstyle\bigodot^2\Lambda^1(2)) \xrightarrow{\mathcal{X}}$$

Killing operator

R. Michel (1973)

$$0 \to \overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet} \to \Gamma(\mathbb{RP}_n, \textstyle\bigodot^2\Lambda^1(4)) \xrightarrow{\nabla} \Gamma(\mathbb{RP}_n, \textstyle\bigodot^3\Lambda^1(4)) \xrightarrow{\mathcal{X}}$$

first BGG operator

P. Estezet (1988)

$$0 \to \overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet} \to \Gamma(\mathbb{RP}_n, \overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}) \xrightarrow{\nabla^{a+1}} \Gamma(\mathbb{RP}_n, \overset{-a-2}{\times}\!-\!\overset{a+b+1}{\bullet}\!-\!\overset{c}{\bullet}) \xrightarrow{\mathcal{X}}$$

first BGG operator

END OF PART TWO

THANK YOU