The X-ray transform: part III

Michael Eastwood

$$\begin{bmatrix} \text{Toby Bailey} & \text{Robin Graham} & \text{Hubert Goldschmidt} \\ \text{Lionel Mason} & \text{Rod Gover} & \text{Laurent Stolovitch} \end{bmatrix}$$

Eduard Čech Institute

Recall X-ray kernels (from Srní 1996)

For all $a, b, c \in \mathbb{Z}_{\geq 0}$, there is an X-ray transform

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \overset{-a-2}{\times}\!\!-\!\!-\!\!\overset{a+b+1}{\bullet}\!\!-\!\!-\!\!\overset{c}{\bullet}) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \text{❄❄❄})$$

and exact sequences

$$0 \to \overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet} \to \Gamma(\mathbb{RP}_3, \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}) \xrightarrow{\nabla^{a+1}} \ker \mathcal{X}$$

first BGG operator

$\overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}$

irreducible

$\mathrm{SL}(4, \mathbb{R})$-module

Example $a = b = c = 0$

For ω a smooth 1-form on $\mathbb{RP}_3$ $\boxed{\text{TFAE}}$

- $\displaystyle\oint_\gamma \omega = 0 \ \forall \gamma$
- $\omega = df$
- $d\omega = 0$ (since $H^1(\mathbb{RP}_3, \mathbb{R}) = 0$)

Some specific kernels

Example $a = 0,\ b = 1,\ c = 0$

For h_{ab} a smooth symmetric tensor on $\mathbb{RP}_3$ $\boxed{\text{TFAE}}$

- $\displaystyle\oint_\gamma h_{ab} = 0\ \forall\,\gamma$
- $h_{ab} = \nabla_{(a}X_{b)}$
- ???

$$\Big[X^a \mapsto \nabla_{(a}X_{b)} \quad \text{Killing} \quad \overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} = \Lambda^2\mathbb{R}^4 = \mathfrak{so}(4)\Big]$$

Example $a = 1,\ b = 0,\ c = 1$

For smooth $\Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c$ on $\mathbb{RP}_3$ s.t. $\Gamma_{ab}{}^b = 0$ $\boxed{\text{TFAE}}$

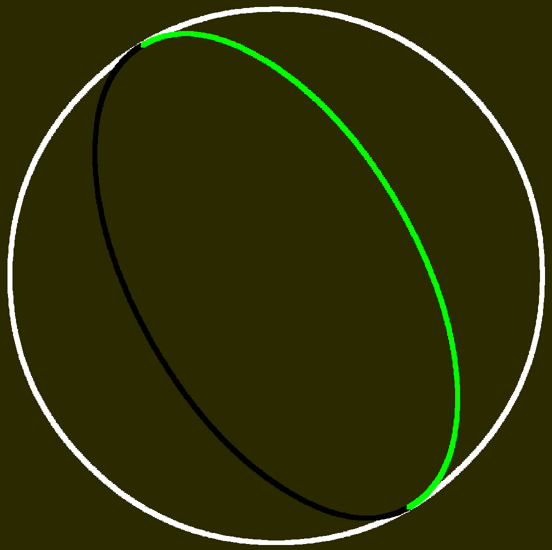
- $\displaystyle\oint_\gamma \Gamma_{ab}{}^c = 0\ \forall\,\gamma$
- $\Gamma_{ab} = (\nabla_{(a}\nabla_{b)}X^c + g_{ab}X^c)_\circ$
- ???

$$\Big[X^a \mapsto (\nabla_{(a}\nabla_{b)}X^c + g_{ab}X^c)_\circ \quad \text{projective Killing}$$

$$\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} = \big((\mathbb{R}^4)^* \otimes \mathbb{R}^4\big)_\circ = \mathfrak{sl}(4,\mathbb{R})\Big]$$

Blaschke conjecture/theorem

On a sphere



- all geodesics are closed
- all geodesics have the same length

The same features are present (in any dimension) on

- real projective space
- complex projective space

CROSSes

Blaschke rigidity

Deformations

- Riemannian $g_{ab} \longmapsto \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$
- Projective $\nabla_a \longmapsto \tilde{\nabla}_a = \nabla_a + \epsilon \Gamma_a$ (more soon...)

Two-sphere with round metric g_{ab}

WLG $\tilde{g}_{ab} = (1+\epsilon f)^2 g_{ab}$

$$\oint_\gamma f = 0 \quad \forall \text{ great circles } \gamma$$

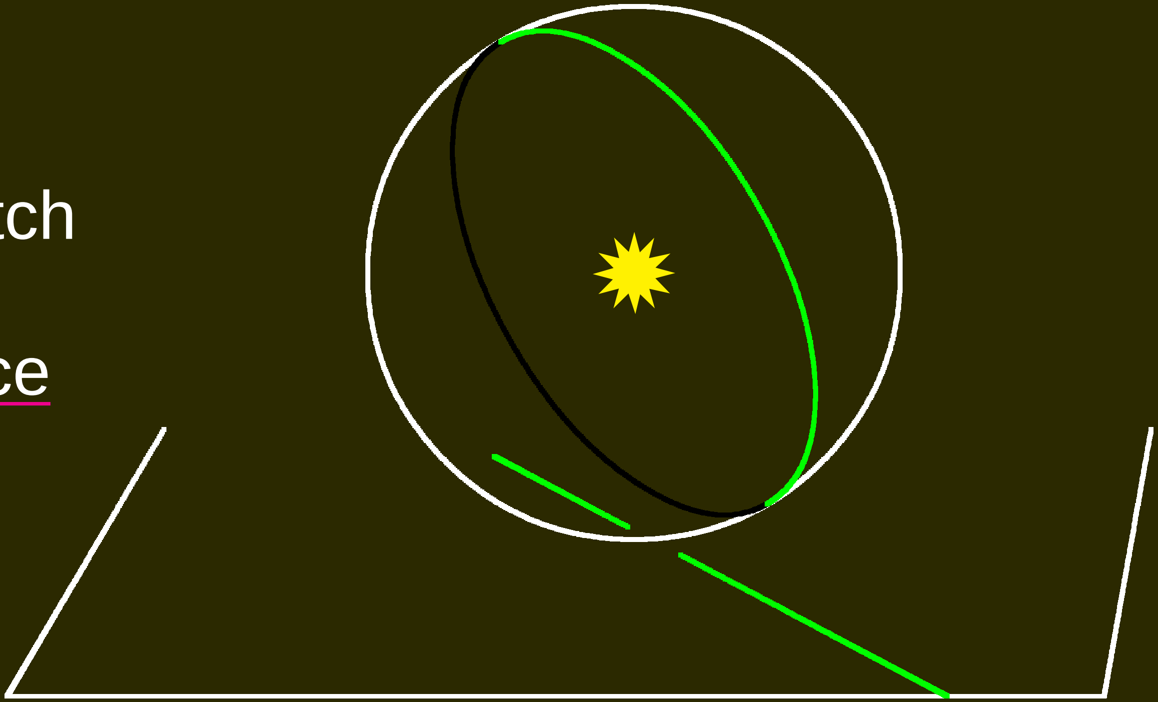
Funk 1914 $\boxed{\oint_\gamma f = 0 \; \forall \gamma \iff f \text{ is odd}}$

$\rightsquigarrow$ Expect $\begin{cases} S^2 \text{ is Blaschke deformable } (\checkmark \text{ Guillemin 1976}) \\ \mathbb{RP}_2 \text{ is Blaschke rigid } (\checkmark \ \ldots \text{ LeBrun–Mason 2002}) \end{cases}$

Projective differential geometry

Def$^{\mathrm{n}}$ $\hat{\nabla}_a \sim \nabla_a \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) the round sphere is projectively flat



Affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a projective equivalence

Operational Def$^{\mathrm{n}}$

$$\boxed{\hat{\nabla}_a\phi_b = \nabla_a\phi_b - \Upsilon_a\phi_b - \Upsilon_b\phi_a}$$

Projective deformations

Projective equivalence

$$\hat{\nabla}_a X^c = \nabla_a X^c + \Gamma_{ab}{}^c X^b \quad \text{where } \Gamma_{ab}{}^c = \Upsilon_a \delta_b{}^c + \Upsilon_b \delta_a{}^c$$

Projective deformation

$$\tilde{\nabla}_a X^c = \nabla_a X^c + \epsilon \Gamma_{ab}{}^c X^b \quad \text{where } \Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c \text{ and } \Gamma_{ab}{}^a = 0$$

Projective deformation complex on S^n or $\mathbb{RP}_n$

$$\begin{array}{ccccccc} X^a & \mapsto & (\nabla_{(a}\nabla_{b)} X^c + g_{ab} X^c)_\circ & & W_{abc}{}^d & \mapsto & \cdots \\ & & \Gamma_{ab}{}^c & \mapsto & (\nabla_{[a}\Gamma_{b]c}{}^d)_\circ & & \end{array}$$

(where $R_{ab}{}^c{}_d = \delta_a{}^c g_{bd} - \delta_b{}^c g_{ad}$ (round metric))

Riemannian deformations

Start with the round metric $R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$

Recall Riemannian deformation

$$\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$$

Riemannian deformation complex on S^n or $\mathbb{RP}_n$

$$X_a \mapsto \nabla_{(a}X_{b)} \qquad ✤ \qquad R_{abcd} \mapsto \nabla_{[a}R_{bc]de}$$

Killing

$$\begin{aligned} h_{ab} \mapsto {} & (\nabla_{(a}\nabla_{c)} + g_{ac})h_{bd} \\ & - (\nabla_{(b}\nabla_{c)} + g_{bc})h_{ad} \\ & - (\nabla_{(a}\nabla_{d)} + g_{ad})h_{bc} \\ & + (\nabla_{(b}\nabla_{d)} + g_{bd})h_{ac} \end{aligned}$$

Bianchi

Riemann tensor symmetries are SL-irreducible !

Projectively invariant complex !

Differential Complexes on $\mathbb{RP}_3$

de Rham

$$0 \to \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \to 0$$

Riemannian deformation

$$0 \to \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \xrightarrow{\nabla} \overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \to 0$$

Projective deformation

$$0 \to \overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \xrightarrow{\nabla^2} \overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \xrightarrow{\nabla^2} \overset{-6}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet} \to 0$$

Killing operator in flat space

Killing operator: $\boxed{X_a \mapsto \nabla_{(a}X_{b)}}$

Solve in flat space: $K_{ab} \equiv \nabla_a X_b$ is skew.

Claim: $\nabla_a K_{bc} = 0$.

$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= 0, \text{ as required.} \end{aligned}$$

Hence, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{array}{rcl} \nabla_a X_b & = & K_{ab} \\ \nabla_a K_{bc} & = & 0 \end{array}}$ closed!

Conclusion: $\boxed{X_a = m_{ab}x^b + r_a}$ where $m_{ab} = -m_{ba}$.

rotations

translations

Curved prolongation

$\nabla_{(a}X_{b)} = 0 \iff \nabla_a X_b = K_{ab}$ for $K_{ab} = K_{[ab]}$.

But then,
$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= R_{bc}{}^d{}_a X_d. \end{aligned}$$

Therefore, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{aligned} \nabla_a X_b &= K_{ab} \\ \nabla_a K_{bc} &= R_{bc}{}^d{}_a X_d \end{aligned}}$

Hence, Killing fields ↔ covariant constant sections of $V \equiv \Lambda^1 \oplus \Lambda^2$ with connection

$$\left[\begin{array}{c} X_b \\ K_{bc} \end{array}\right] \xmapsto{\nabla_a} \left[\begin{array}{c} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} - R_{bc}{}^d{}_a X_d \end{array}\right].$$

Curvature

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix} X_c \\ K_{cd} \end{bmatrix} = \begin{bmatrix} 0 \quad \text{(by design)} \\ \underbrace{R \bowtie K}_{\boxed{2R_{ab}{}^{e}{}_{[c}K_{d]e} + 2R_{cd}{}^{e}{}_{[a}K_{b]e}}} + \underbrace{(\nabla R) \bowtie X}_{\boxed{(\nabla_b R_{cd}{}^{e}{}_{a})X_e - (\nabla_a R_{cd}{}^{e}{}_{b})X_e}} \end{bmatrix}$$

Flat $\iff$ $R_{abcd} = \lambda(g_{ac}g_{bd} - g_{bc}g_{ad})$

$\iff$ constant curvature.

Sphere or $\mathbb{RP}_n$ has symmetries of maximal dimension

$$\dim\Lambda^1 + \dim\Lambda^2 = n + n(n-1)/2 = \text{n(n+1)/2}$$

$$= \dim \mathrm{SO}(n+1) = \dim \mathfrak{so}(n+1).$$

Coupled de Rham sequence

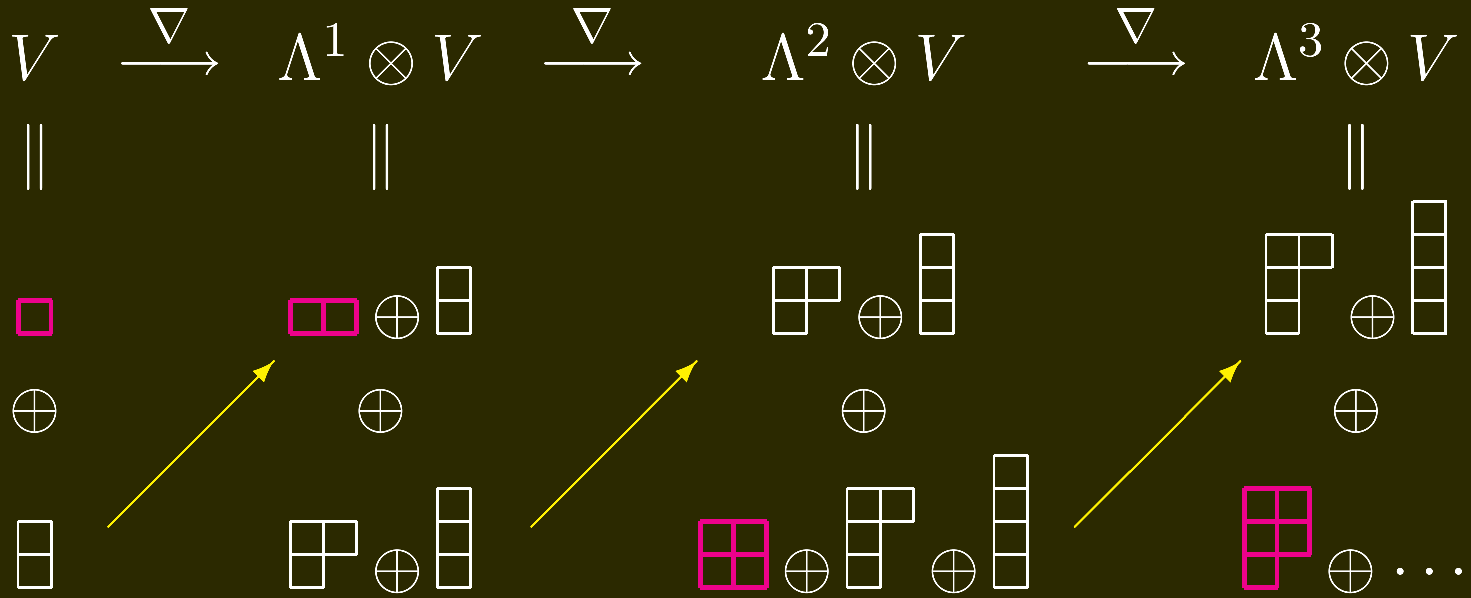
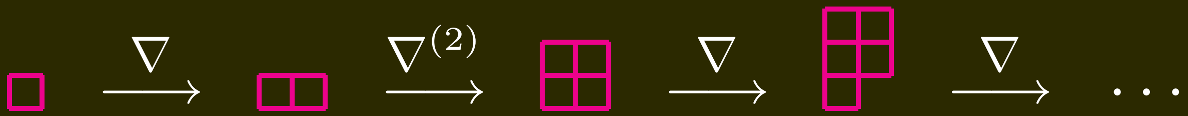
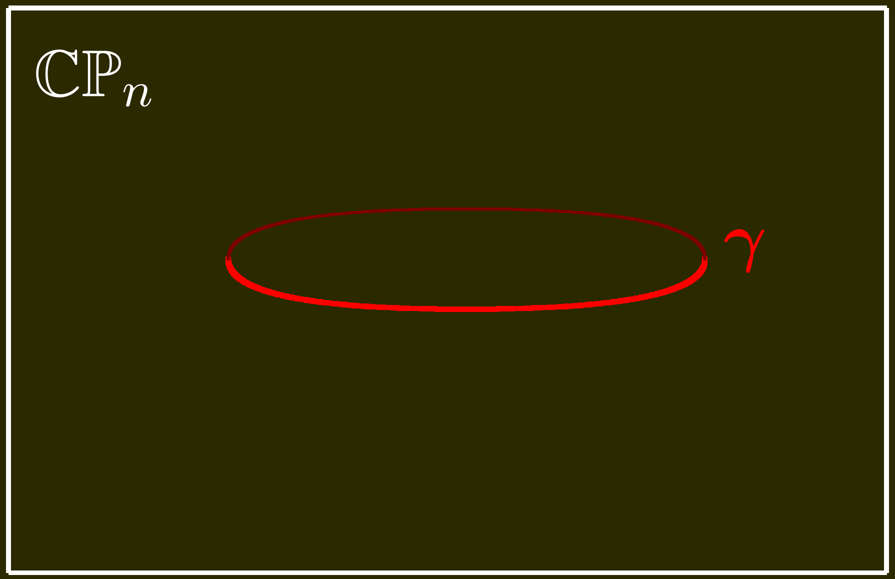


Diagram chasing in the constant curvature case ⇝



a locally exact complex!

X-ray transform on $\mathbb{CP}_n$



$\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$

Fubini-Study metric

$f =$ smooth function on $\mathbb{CP}_n$

$\gamma =$ geodesic

$$f \xmapsto{\mathcal{X}} \phi(\gamma) = \oint_\gamma f$$

Questions

- Kernel of $\mathcal{X}$?
- What about $\omega \xmapsto{\mathcal{X}} \phi(\gamma) = \oint_\gamma \omega$ for ω a 1-form?
- What about $h_{ab\cdots c}$ a symmetric tensor?

X-ray transform on functions

Know $\oint_\gamma f = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \;\Leftrightarrow\; f = 0.$

Suppose

Funk (1913)

$$\oint_\gamma f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

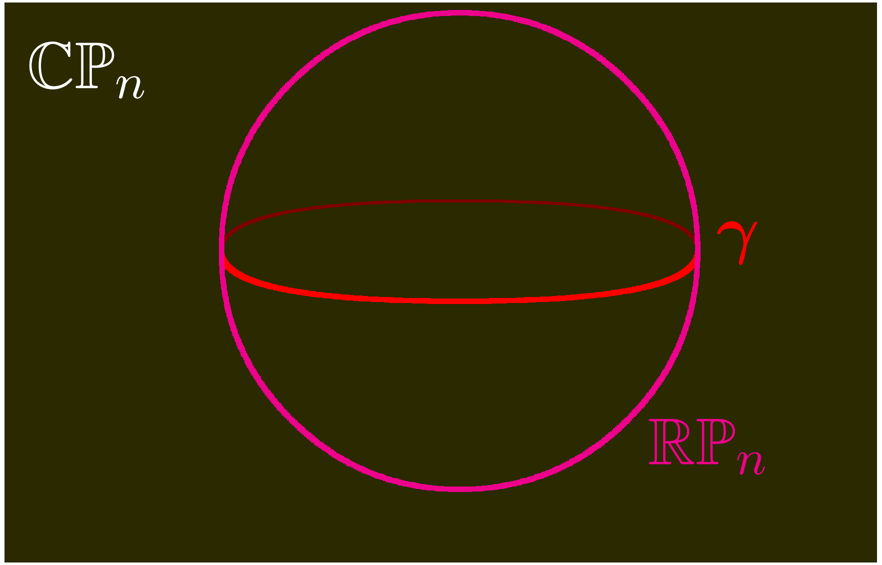
Then $\oint_\gamma f = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \overset{\mu}{\hookrightarrow} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* f = 0 \;\; \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence $f = 0$, i.e. $\boxed{\mathcal{X} \text{ is injective on functions on } \mathbb{CP}_n}$

cf. Helgason, The Radon Transform, §2 Corollary 2.3

Approach (with Hubert Goldschmidt)



$\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ induced by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is totally geodesic.

Translates by $\mathrm{SU}(n+1)$ too!

↑ ‘Model Embeddings’ μ

- The X-ray transform on $\mathbb{RP}_n$ is well-understood.
- Pullback of tensors under μ is well-understood.
- Suitable global techniques on $\mathbb{CP}_n$ are available,
- compatible with similar techniques (BGG) on $\mathbb{RP}_n$.

X-ray transform on 1-forms

Know $\quad \oint_\gamma \omega = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \quad \Leftrightarrow \quad \omega = d\phi.$

Suppose Michel (1978)

$$\oint_\gamma \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_\gamma \omega = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \overset{\mu}{\hookrightarrow} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* d\omega = 0 \;\; \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence, by 2-form lemma, $\boxed{d\omega = \theta J}$. Claim $\boxed{\omega = d\phi}$

1-forms cont'd

Theorem For $\omega \in \Gamma(\mathbb{CP}_n, \Lambda^1),\ n \geq 2$, TFAE

(a) $\omega = d\phi$

(b) $d\omega = 0$

(c) $(d\omega)_\perp = 0$

(d) $d\omega = \theta J$

Proof (a)⇔(b) because $H^1(\mathbb{CP}_n, \mathbb{R}) = 0$.
(c)⇔(d) by definition.
(b)⇒(c) is trivial.

(d)⇒(b) If $d\omega = \theta J$, then

$$0 = d^2\omega = d(\theta J) = d\theta \wedge J \Rightarrow d\theta = 0 \Rightarrow \theta = \text{constant.}$$

But if $\theta \neq 0$, then $d(\omega/\theta) = J$, a contradiction. □

X-ray transform on $\mathbb{CP}_n$

Warning ☠↯☠↯☠ $\mathbb{CP}_n$ is not projectively flat !

$$W_{ab}{}^{c}{}_{d} = 2J_{[a}{}^{c}J_{b]d} - 2J_{ab}J_{d}{}^{c} - \tfrac{6}{2n-1}\delta_{[a}{}^{c}g_{b]d}$$

$$\boxed{\Gamma(\mathbb{CP}_n, \bigodot^{m-1}\Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \bigodot^{m}\Lambda^1) \ni h \xmapsto{I_m} \oint_\gamma h}$$

Theorem (Tsukamoto 1981) $\ker I_2 = \operatorname{range} \nabla$

Blaschke rigidity

Theorem (E-Goldschmidt 2013) $\ker I_m = \operatorname{range} \nabla$

Curvature on $\mathbb{RP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

Curvature on $\mathbb{CP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Killing operator on $\mathbb{CP}_n$

Tractor onnection on $\Lambda^1 \oplus \Lambda^2$:–

$$\left[\begin{array}{l} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b - J_{ab} J_c{}^d X_d + J_{ac} J_b{}^d X_d + 2 J_{bc} J_a{}^d X_d \end{array}\right]$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \end{array}\right] = 4 \left[\begin{array}{c} 0 \\ J_{c[a} \tilde{K}_{b]d} - J_{d[a} \tilde{K}_{b]c} - J_{ab} \tilde{K}_{cd} - J_{cd} \tilde{K}_{ab} \end{array}\right]$$

where $\tilde{K}_{ab} \equiv J_{[a}{}^e K_{b]e}$.

Better connection

Better connection on $\Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$:–
(cf. 'conformally Fedosov manifolds' with Jan Slovák)

$$\left[\begin{array}{c} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b + J_{ab} L_c - J_{ac} L_b - J_{bc} L_a + J_{bc} J_a{}^d X_d \\ \nabla_a L_b + J_a{}^d K_{bd} \end{array}\right]$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \\ L_c \end{array}\right] = 2J_{ab} \left[\begin{array}{c} L_c + J_c{}^e X_e \\ -J_c{}^e K_{de} + J_d{}^e K_{ce} \\ -X_c + J_c{}^e L_e \end{array}\right]$$

Better coupling

$$\begin{tikzcd}
V \arrow[rr, "\nabla"] \arrow[d, equal] & & \Lambda^1 \otimes V \arrow[rr, "\nabla_\perp"] \arrow[d, equal] & & \Lambda^2_\perp \otimes V \arrow[d, equal] \\
\ydiagram{1} & & \ydiagram{2} \oplus \ydiagram{1,1}_\perp \oplus \mathbb{R} & & \ydiagram{2,1}_\perp \oplus \ydiagram{1,1,1}_\perp \oplus \ydiagram{1} \\
\oplus & & \oplus & & \oplus \\
\ydiagram{1,1}_\perp \oplus \mathbb{R} \arrow[uurr] & & \ydiagram{2,1}_\perp \oplus \ydiagram{1,1,1}_\perp \oplus 2\,\ydiagram{1} \arrow[uurr] & & \ydiagram{2,2}_\perp \oplus 6\,\text{more} \\
\oplus & & \oplus & & \oplus \\
\ydiagram{1} \arrow[uurr] & & \ydiagram{2} \oplus \ydiagram{1,1}_\perp \oplus \mathbb{R} \arrow[uurr] & & \ydiagram{2,1}_\perp \oplus \ydiagram{1,1,1}_\perp \oplus \ydiagram{1}
\end{tikzcd}$$

cf. Andreas Čap's talk

(from Lie algebra cohomology (Heisenberg algebra))

$$h_{ab} = \nabla_{(a}X_{b)} \Leftrightarrow \boxed{\pi_\perp(\nabla_{(a}\nabla_{c)}h_{bd} + g_{ac}h_{bd}) = 0}$$

$\perp =$ trace free part w.r.t. J_{ab}, e.g. $\ydiagram{2,2} = \ydiagram{2,2}_\perp \oplus \ydiagram{1,1}_\perp \oplus \mathbb{R}$

THE END

THANK YOU