The Projective Rigidity of Projective Space

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[background to joint work with Laurent Stolovitch]

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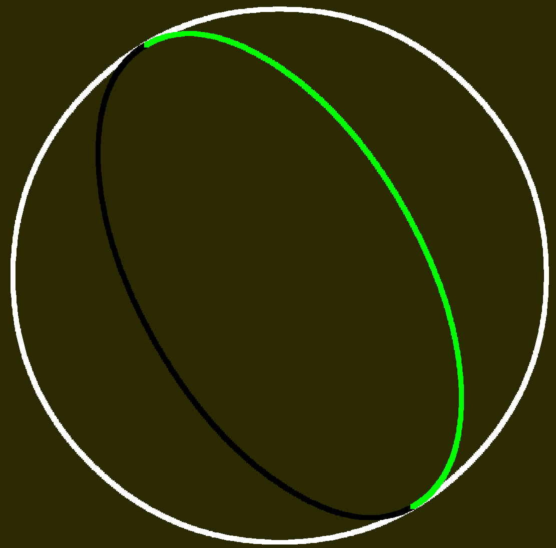
Blaschke conjecture/theorem

On a sphere



- all geodesics are closed
- all geodesics have the same length

The same features are present (in any dimension) on

- real projective space
- complex projective space

CROSSes

Blaschke rigidity

Deformations

- Riemannian $g_{ab} \longmapsto \tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$
- Projective $\nabla_a \longmapsto \tilde{\nabla}_a = \nabla_a + \epsilon \Gamma_a$

Two-sphere with round metric g_{ab}

WLG $\quad \tilde{g}_{ab} = (1 + \epsilon f)^2 g_{ab}$

$$\oint_\gamma f = 0 \quad \forall \text{ great circles } \gamma$$

Funk 1914 $\quad \boxed{\oint_\gamma f = 0 \;\forall \gamma \iff f \text{ is odd}}$ (cf. Radon 1917)

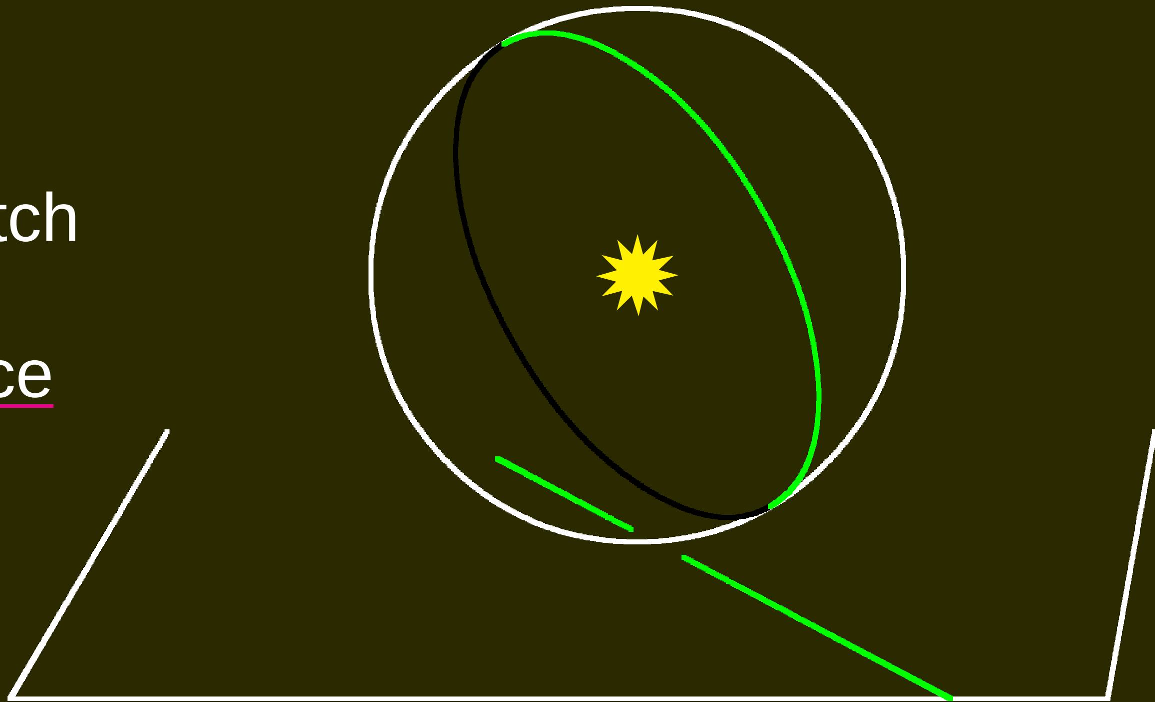
$\rightsquigarrow$ Expect $\begin{cases} S^2 \text{ is Blaschke deformable } (\checkmark \text{ Guillemin 1976}) \\ \mathbb{RP}_2 \text{ is Blaschke rigid } (\checkmark \ldots \text{ LeBrun–Mason 2002}) \end{cases}$

Projective differential geometry

Def$^{\mathrm{n}}$ $\hat{\nabla}_a \sim \nabla_a \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) the round sphere is projectively flat

Affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a projective equivalence



Operational Def$^{\mathrm{n}}$

$$\hat{\nabla}_a\phi_b = \nabla_a\phi_b - \Upsilon_a\phi_b - \Upsilon_b\phi_a$$

Projective deformations

Projective equivalence

$$\hat{\nabla}_a X^c = \nabla_a X^c + \Gamma_{ab}{}^c X^b \quad \text{where } \Gamma_{ab}{}^c = \Upsilon_a \delta_b{}^c + \Upsilon_b \delta_a{}^c$$

Projective deformation

$$\tilde{\nabla}_a X^c = \nabla_a X^c + \epsilon \Gamma_{ab}{}^c X^b \quad \text{where } \Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c \text{ and } \Gamma_{ab}{}^a = 0$$

Projective deformation complex on S^n or $\mathbb{RP}_n$

$$X^a \mapsto (\nabla_{(a}\nabla_{b)} X^c + g_{ab} X^c)_\circ \qquad W_{abc}{}^d \mapsto \cdots$$

$$\Gamma_{ab}{}^c \mapsto (\nabla_{[a}\Gamma_{b]c}{}^d)_\circ$$

(where $R_{abc}{}^d = g_{ac}\delta_b{}^d - g_{bc}\delta_a{}^d$ (round metric))

Riemannian deformations

Start with the round metric $R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$

Recall Riemannian deformation

$$\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$$

Riemannian deformation complex on S^n or $\mathbb{RP}_n$

$$X_a \mapsto \nabla_{(a}X_{b)} \qquad R_{abcd} \mapsto \nabla_{[a}R_{bc]de}$$

Killing

$$\begin{aligned} h_{ab} \mapsto\ & (\nabla_{(a}\nabla_{c)} + g_{ac})h_{bd} \\ & - (\nabla_{(b}\nabla_{c)} + g_{bc})h_{ad} \\ & - (\nabla_{(a}\nabla_{d)} + g_{ad})h_{bc} \\ & + (\nabla_{(b}\nabla_{d)} + g_{bd})h_{ac} \end{aligned}$$

Bianchi

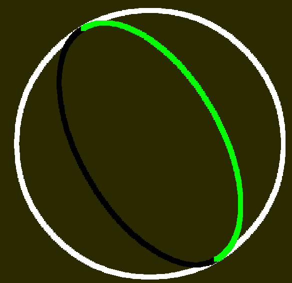
Riemann tensor symmetries are SL-irreducible !

Projectively invariant complex !

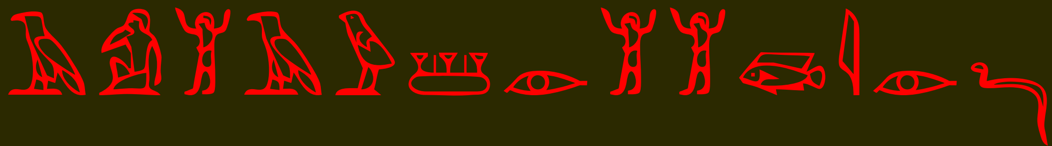
Homogeneous correspondence

Homogeneous under action of $\mathrm{SL}(n+1,\mathbb{R})$

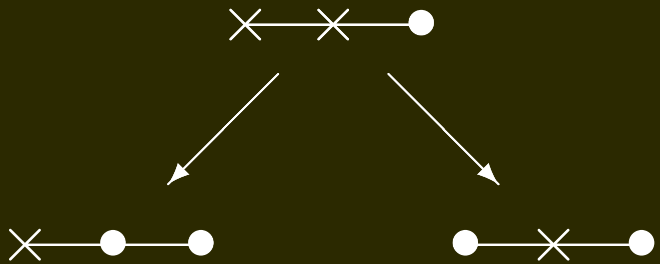
$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) & \\ \swarrow\mu & & \nu\searrow \\ \mathbb{RP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \ni x \rightsquigarrow \gamma = \mu(\nu^{-1}(x))\end{array}$$



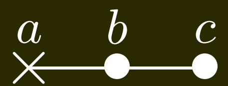
Hieroglyphics



$n = 3$



Homogeneous vector bundles



Differential Complexes on $\mathbb{RP}_3$

de Rham

$$0 \to \overset{0}{\times}{-}\overset{0}{\bullet}{-}\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}{-}\overset{1}{\bullet}{-}\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-3}{\times}{-}\overset{0}{\bullet}{-}\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}{-}\overset{0}{\bullet}{-}\overset{0}{\bullet} \to 0$$

Riemannian deformation

$$0 \to \overset{0}{\times}{-}\overset{1}{\bullet}{-}\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}{-}\overset{2}{\bullet}{-}\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-4}{\times}{-}\overset{0}{\bullet}{-}\overset{2}{\bullet} \xrightarrow{\nabla} \overset{-5}{\times}{-}\overset{0}{\bullet}{-}\overset{1}{\bullet} \to 0$$

Projective deformation

$$0 \to \overset{1}{\times}{-}\overset{0}{\bullet}{-}\overset{1}{\bullet} \xrightarrow{\nabla^2} \overset{-3}{\times}{-}\overset{2}{\bullet}{-}\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}{-}\overset{1}{\bullet}{-}\overset{2}{\bullet} \xrightarrow{\nabla^2} \overset{-6}{\times}{-}\overset{1}{\bullet}{-}\overset{0}{\bullet} \to 0$$

Generalised Funk transform on $\mathbb{RP}_3$

General complex

$$0 \to \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet} \xrightarrow{\nabla^{a+1}} \overset{-a-2}{\times}\!\!-\!\!\overset{a+b+1}{\bullet}\!\!-\!\!\overset{c}{\bullet} \longrightarrow \overset{-a-b-3}{\times}\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\overset{b+c+1}{\bullet} \longrightarrow \overset{-a-b-c-4}{\times}\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet} \to 0$$

$$\Gamma(\mathbb{RP}_3, \overset{-a-2}{\times}\!\!-\!\!\overset{a+b+1}{\bullet}\!\!-\!\!\overset{c}{\bullet}) \ni f \overset{\mathcal{F}}{\longmapsto} \oint_\gamma f \in \widetilde{\Gamma}(\mathrm{Gr}_2(\mathbb{R}^4), \overset{a}{\bullet}\!\!-\!\!\overset{b}{\times}\!\!-\!\!\overset{c}{\bullet})$$

Theorem (Bailey–E 1997) $\ker \mathcal{F} = \nabla^{a+1}\big(\Gamma(\mathbf{RP}_3, \overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet})\big)$

Examples $\overset{-2}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet} \leftrightsquigarrow$ Metric rigidity

$\overset{-3}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{1}{\bullet} \leftrightsquigarrow$ projective rigidity

$\overset{-2}{\times}\!\!-\!\!\overset{m}{\bullet}\!\!-\!\!\overset{0}{\bullet} \leftrightsquigarrow$ injectivity of I_m on symmetric solenoidal fields

Generalised Funk transform on $\mathbb{CP}_n$

Warning ☠ ↯ ☠ ↯ ☠ $\mathbb{CP}_n$ is not projectively flat !

$$W_{ab}{}^c{}_d = 2J_{[a}{}^c\omega_{b]d} - 2\omega_{ab}J_d{}^c - \frac{6}{2n-1}\delta_{[a}{}^c g_{b]d}$$

$$\Gamma(\mathbb{CP}_n, \bigodot^{m-1}\Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \bigodot^m \Lambda^1) \ni f \xmapsto{I_m} \oint_\gamma f$$

Theorem (Tsukamoto 1981) $\ker I_2 = \operatorname{range} \nabla$

Theorem (E-Goldschmidt 2013) $\ker I_m = \operatorname{range} \nabla$

Technique $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ is totally geodesic and ...

?Conjecture? $\mathbb{CP}_n$ is projectively rigid

THE END

THANK YOU