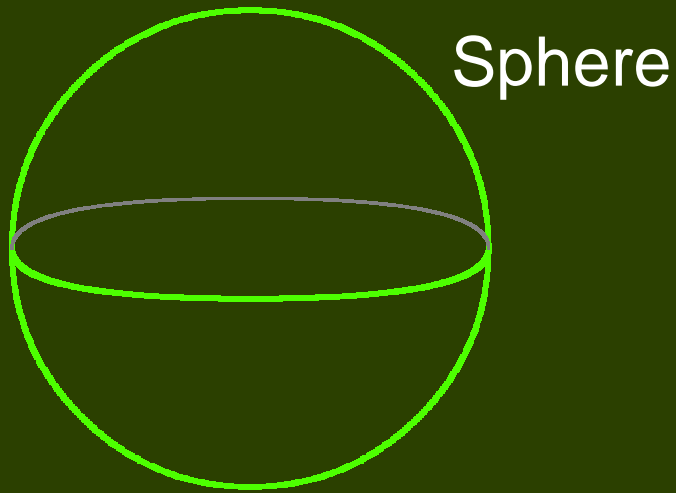




Symmetry and Differential Equations

Michael Eastwood

Symmetry

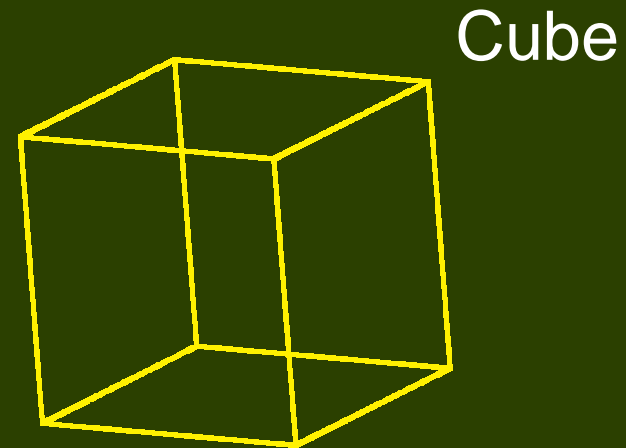


Continuous symmetry

EG: $G = O(3)$ or $SO(3)$

“spherical symmetry,” “spherical polar coordinates,” ...

Lie groups

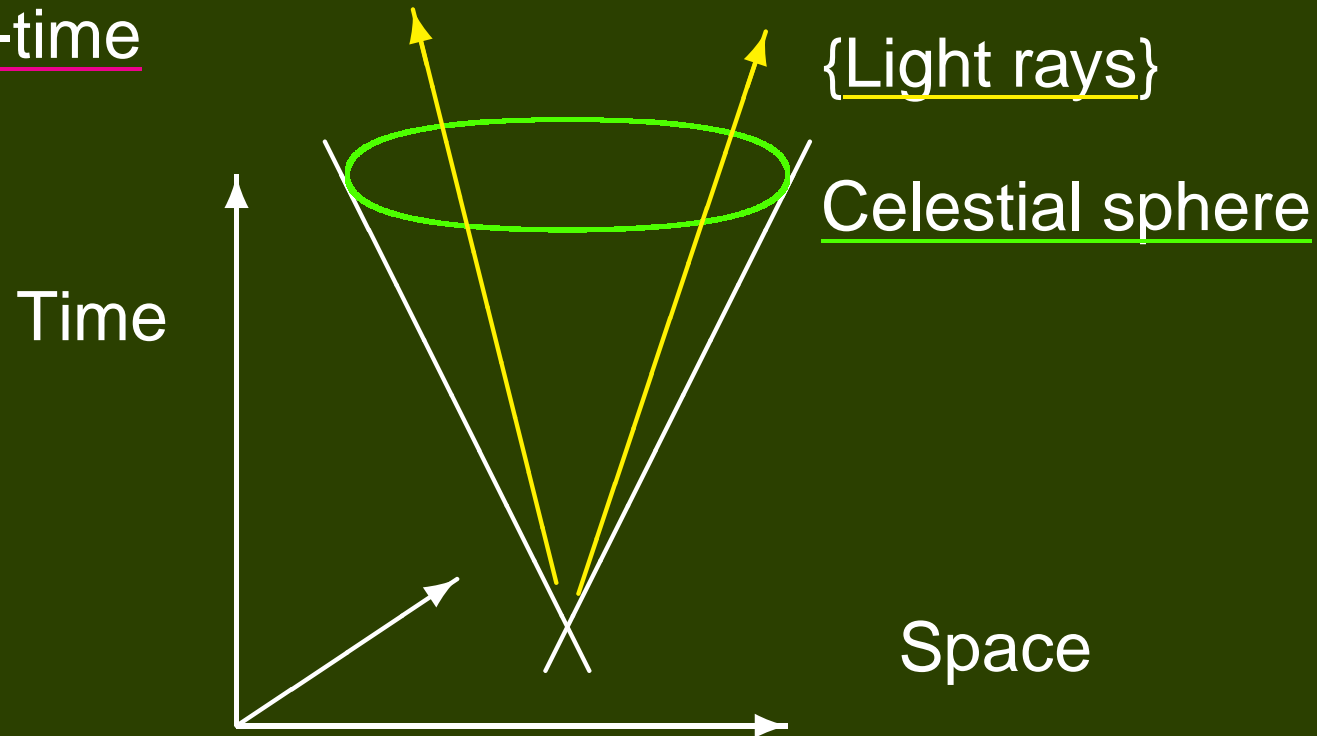


Finite symmetry

$G = S_4 \times C_2$ or S_4

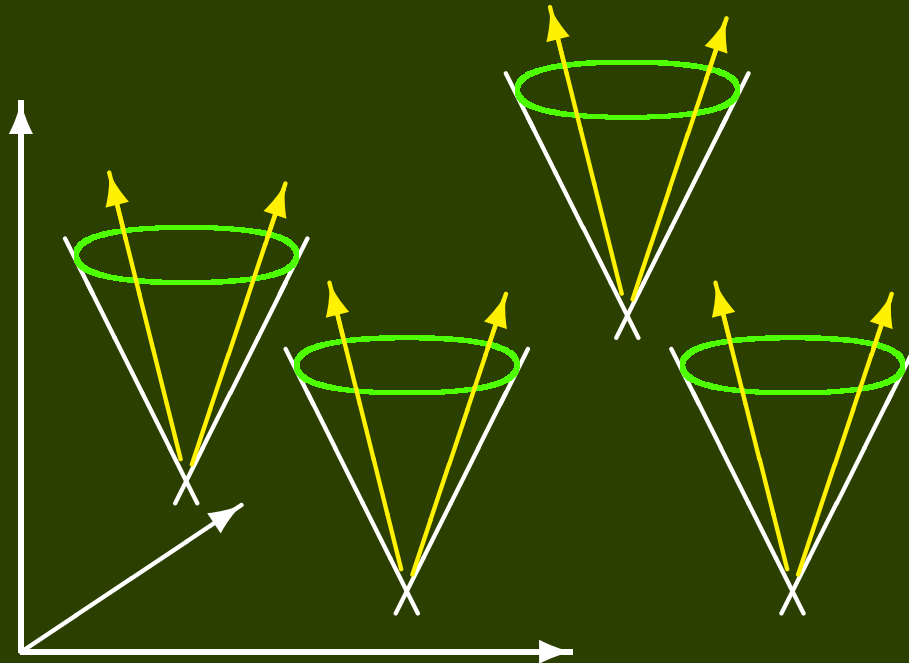
Symmetry cont'd

Space-time



Lorentz group $SO(3, 1)$
also acting on the sphere!

Differential Equations



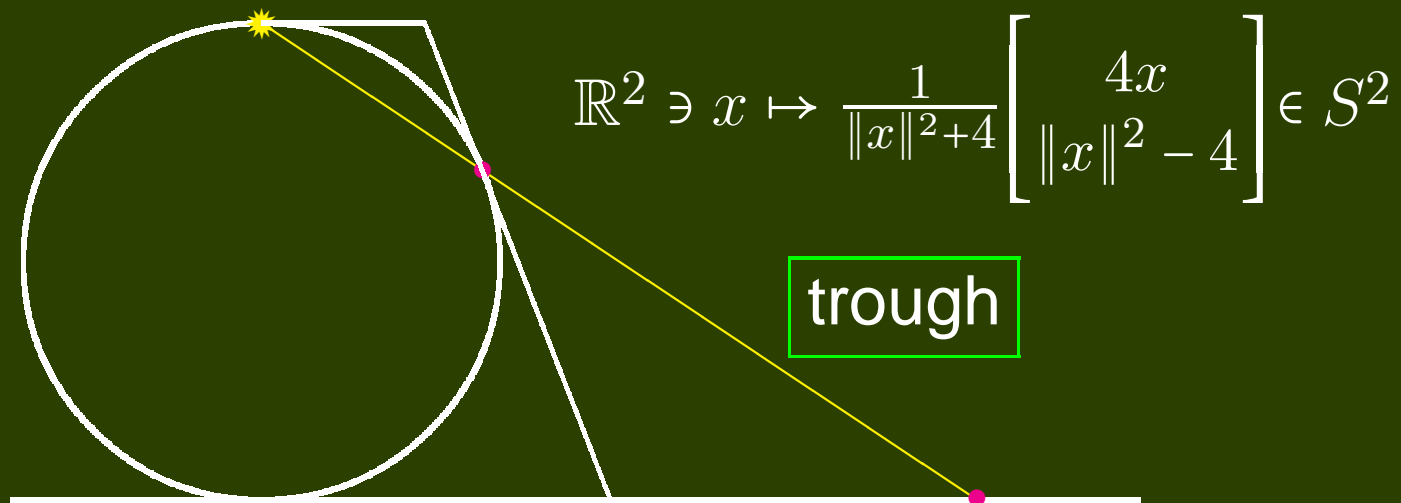
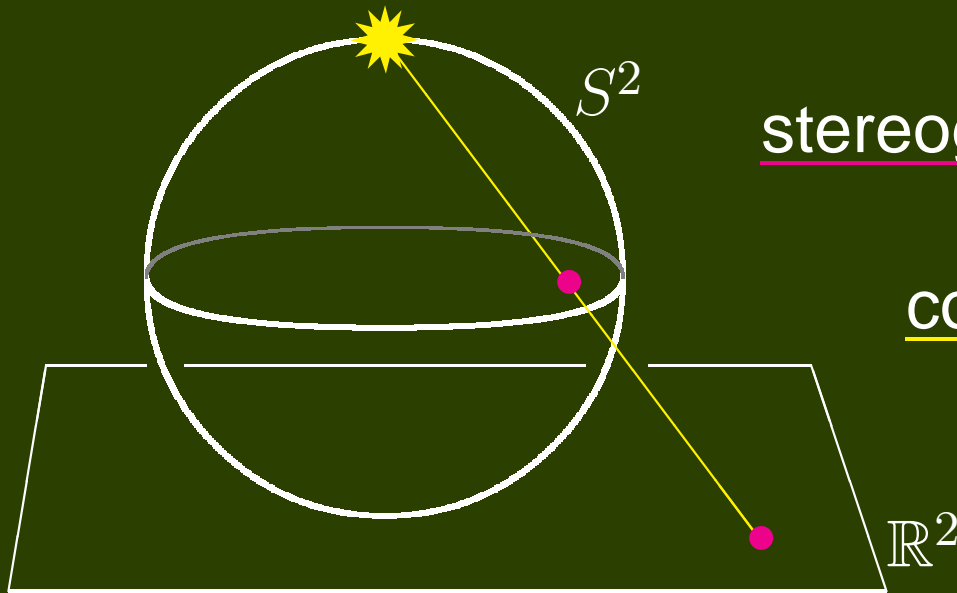
Minkowski, Einstein, ...

Maxwell “Let there be electromagnetic radiation”

Electric field E }
Magnetic field B } \leadsto 2-form F (NB: $3 + 3 = 6 = \dim \Lambda^2 \mathbb{R}^4$)

Maxwell's equations $dF = 0 \quad d*F = 0$

Two-dimensional conformal geometry

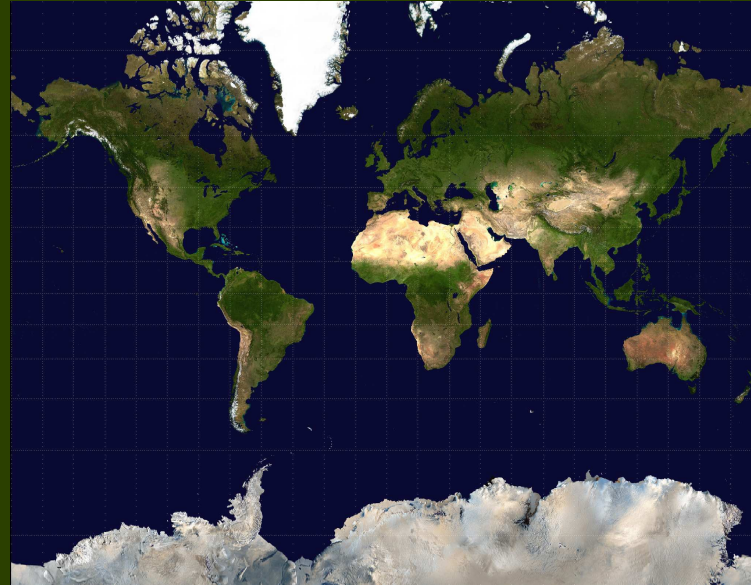


$$\mathbb{R}^2 \ni x \mapsto \frac{1}{\|x\|^2+4} \begin{bmatrix} 4x \\ \|x\|^2 - 4 \end{bmatrix} \in S^2$$

trough

Motivation from navigation

- Mercator
(Cartographer) 1569

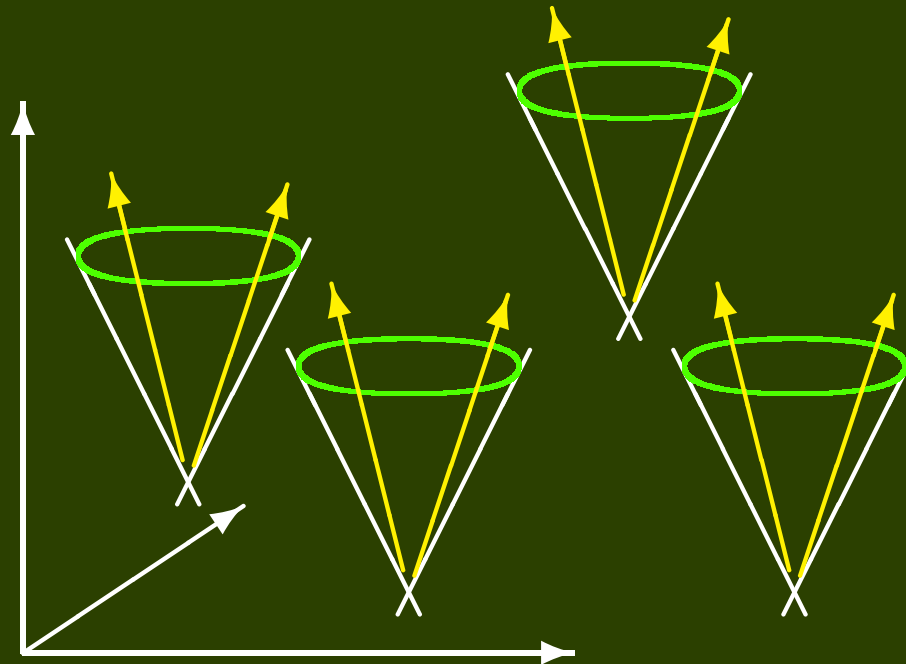


- Wright (Mathematician) 1599

$$S^2 \setminus \{\text{poles}\} \xrightarrow{\text{stereographic}} \mathbb{R}^2 \setminus \{0\} = \mathbb{C} \setminus \{0\} \xrightarrow{\log} \mathbb{C}$$

$$\text{Jac} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \iff \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \quad \boxed{\text{Cauchy-Riemann}}$$

Four-dimensional conformal geometry



$$\text{SO}(3, 1) \cong \text{SL}(2, \mathbb{C})$$

$$\begin{aligned} dF &= 0 \\ d*F &= 0 \end{aligned}$$

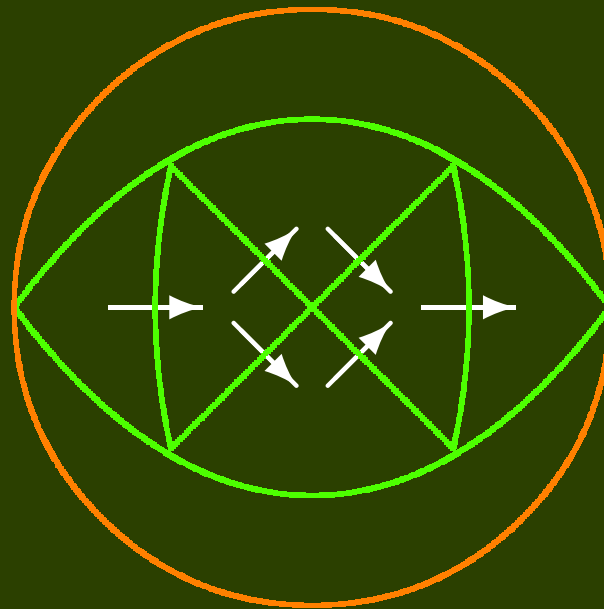
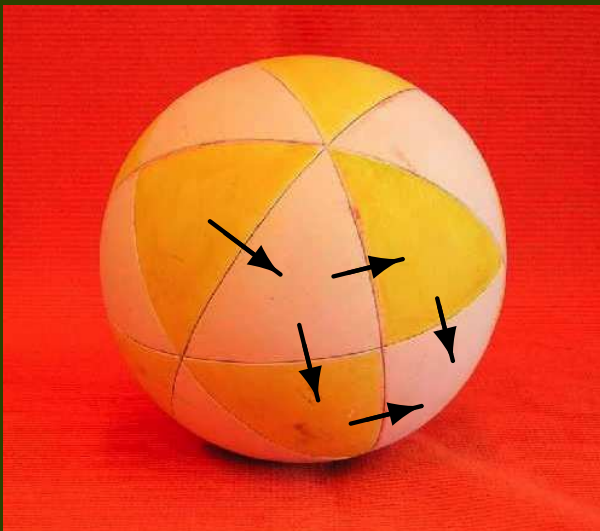
Maxwell's equations are conformally invariant!

$$0 \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \begin{matrix} \nearrow \Lambda^2_+ \\ \searrow \Lambda^2_- \end{matrix} \oplus \Lambda^2 \rightarrow \Lambda^3 \rightarrow \Lambda^4 \rightarrow 0$$

de Rham cont'd

$$0 \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \begin{array}{l} \nearrow \Lambda^2_+ \\ \searrow \Lambda^2_- \end{array} \oplus \begin{array}{l} \nearrow \Lambda^3 \\ \searrow \Lambda^4 \end{array} \rightarrow 0$$

Road map



Can view as
a lune on a sphere!

The countries are
A3 Weyl chambers!

Differential complexes and classification

Parabolic subgroups of $SL(4, \mathbb{R})$

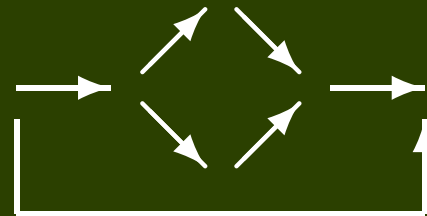
$\xleftrightarrow{1:1}$ lunes compatible with the A3 tiling of the sphere

Parabolic differential geometry



Parabolic differential geometry cont'd

Enhanced road map



Conformal
Laplacian

NB $SL(4, \mathbb{R})$ also acts on the three-sphere

\leadsto projective differential geometry

\leadsto applications in numerical analysis

In the other direction (differential equations \rightarrow symmetries)

Laplacian on $\mathbb{R}^n \leadsto$ conformal symmetries

$\leadsto so(n+1, 1) \leadsto$ Joseph ideal $\subset \mathfrak{U}(so(n+1, 1))$

$\leadsto \dots$



THE END

THANK YOU