Parabolic geometry in five dimensions

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Five-dim$^\ell$ projective geometry

Real projective space

$$\begin{aligned}
\mathbb{RP}_5 &= \mathbb{R}^5 \sqcup \mathbb{RP}_4 \quad \text{(affine 'cell' with points 'at infinity')} \\
&= \frac{\mathbb{R}^6 \setminus \{0\}}{x \sim \lambda x \quad \forall\, \lambda \neq 0} \\
&= \mathrm{SL}(6,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} \lambda & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} \text{ s.t. } \lambda \neq 0 \right\}
\end{aligned}$$

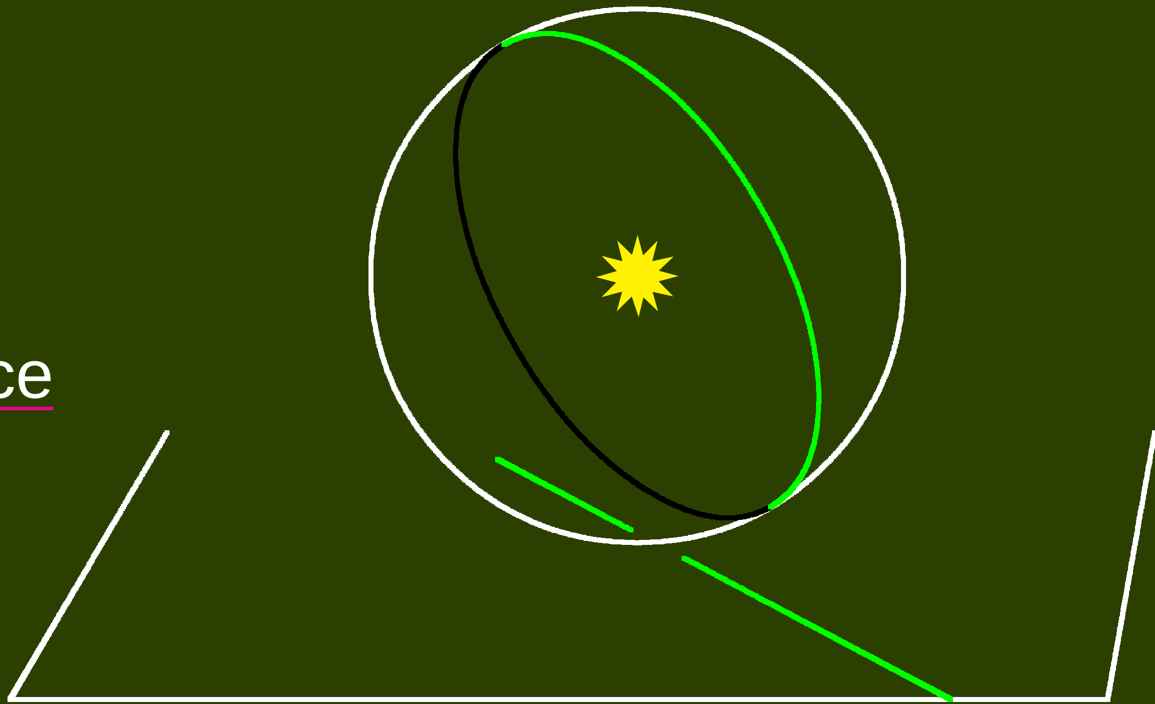
Real projective sphere

$$S^5 = \frac{\mathbb{R}^6 \setminus \{0\}}{x \sim \lambda x \quad \forall\, \lambda > 0} = \mathrm{SL}(6,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} \lambda & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} \text{ s.t. } \lambda > 0 \right\}$$

Five-dim$^\ell$ projective differential geometry

Defⁿ $\hat{\nabla}_a \sim \nabla_a \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) the round sphere is projectively flat



Affine cell
$\mathbb{R}^5 \hookrightarrow \mathbb{RP}_5$ is a
projective equivalence

Projective motions?

induced by $\boxed{\mathrm{SL}(6,\mathbb{R})}$!! NB !!

Five-dim$^\ell$ flat models

Flag manifolds

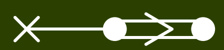
$$\mathrm{SL}(\ell+1,\mathbb{C})\Big/\left\{\begin{pmatrix} * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix}\right\} =$$

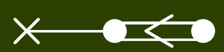
$$/\!/ \quad G/P$$

Dynkin diagram with ℓ nodes

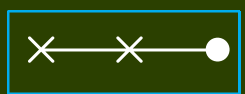
All five-dimensional cases (with G simple)

 projective

 conformal (quadric)

 contact projective

 contact Legendrian (cf. CR geometry)

   (2,3,5) geometry (quadric)

Conformal geometry in four variables

The de Rham complex in four (conformal) variables

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \begin{matrix} \nearrow & \Lambda^2_+ & \searrow \\ & \oplus & \\ \searrow & \Lambda^2_- & \nearrow \end{matrix} \Lambda^3 \longrightarrow \Lambda^4 \to 0$$

$$\left.\begin{array}{l} \Lambda^2_+ = \{\omega \text{ s.t. } *\omega = +\omega\} \\ \Lambda^2_- = \{\omega \text{ s.t. } *\omega = -\omega\} \end{array}\right\}$$ in Riemannian or neutral signature

$\mathrm{SO}^{\uparrow}(2,2) \xleftarrow{1:2} \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ spinors

$\mathrm{S}(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}))$ conformal spinors

Conformal geometry: the flat model

$$M = \mathrm{Gr}_2(\mathbb{R}^4) = \{\Pi \subset \mathbb{R}^4 \text{ s.t. } \dim \Pi = 2\}$$

$$= \mathrm{SL}(4,\mathbb{R}) \Big/ \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} = G/P = \bullet\!\!-\!\!\times\!\!-\!\!\bullet$$

$$\mathrm{S}(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}))$$

$$T_\Pi M = \mathrm{Hom}(\Pi, \mathbb{R}^4/\Pi) = \Pi^* \otimes \mathbb{R}^4/\Pi \rightsquigarrow S' \otimes S = \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} \otimes \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}$$

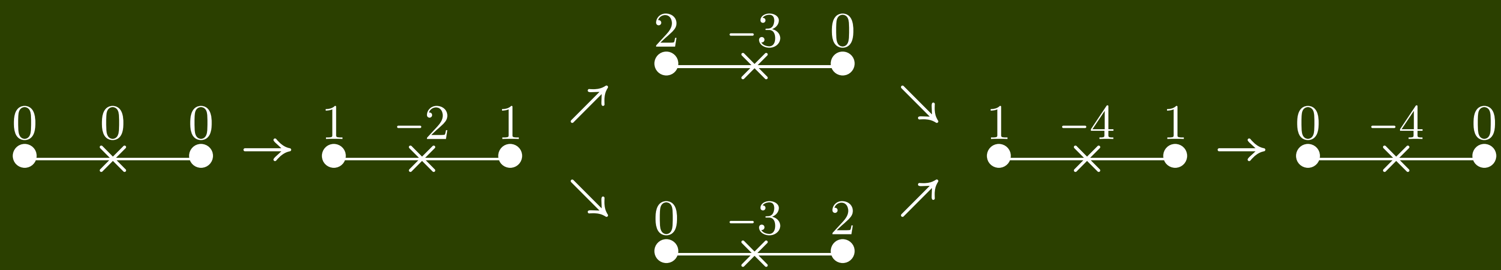
spin bundles

Dually,

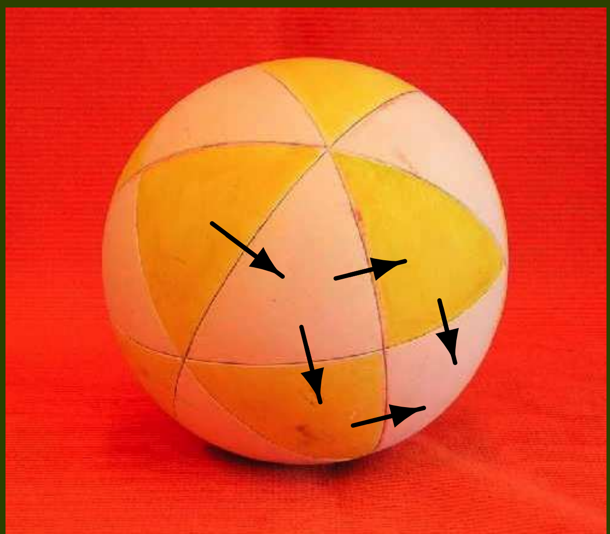
$$\Lambda^1_M = S'^* \otimes S^* = \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \otimes \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet} = \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}$$

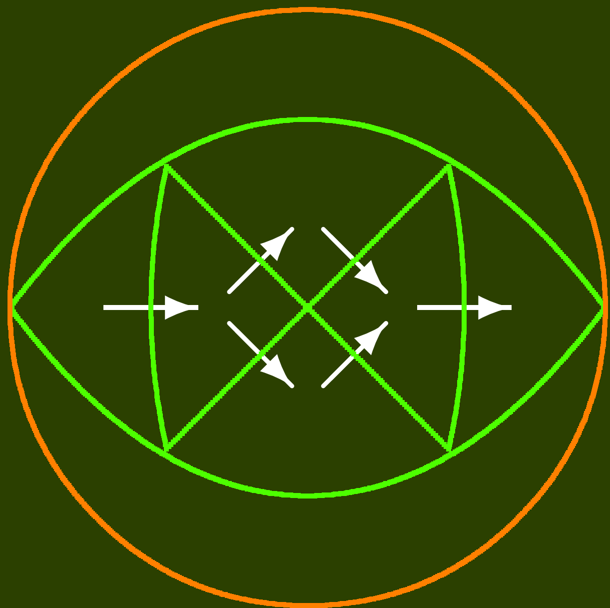
$$\Lambda^2_M = (\bigodot^2 S'^* \otimes \Lambda^2 S^*) \oplus (\Lambda^2 S'^* \otimes \bigodot^2 S^*) = \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet} = \Lambda^2_+ \oplus \Lambda^2_-$$

de Rham revisited



Road map



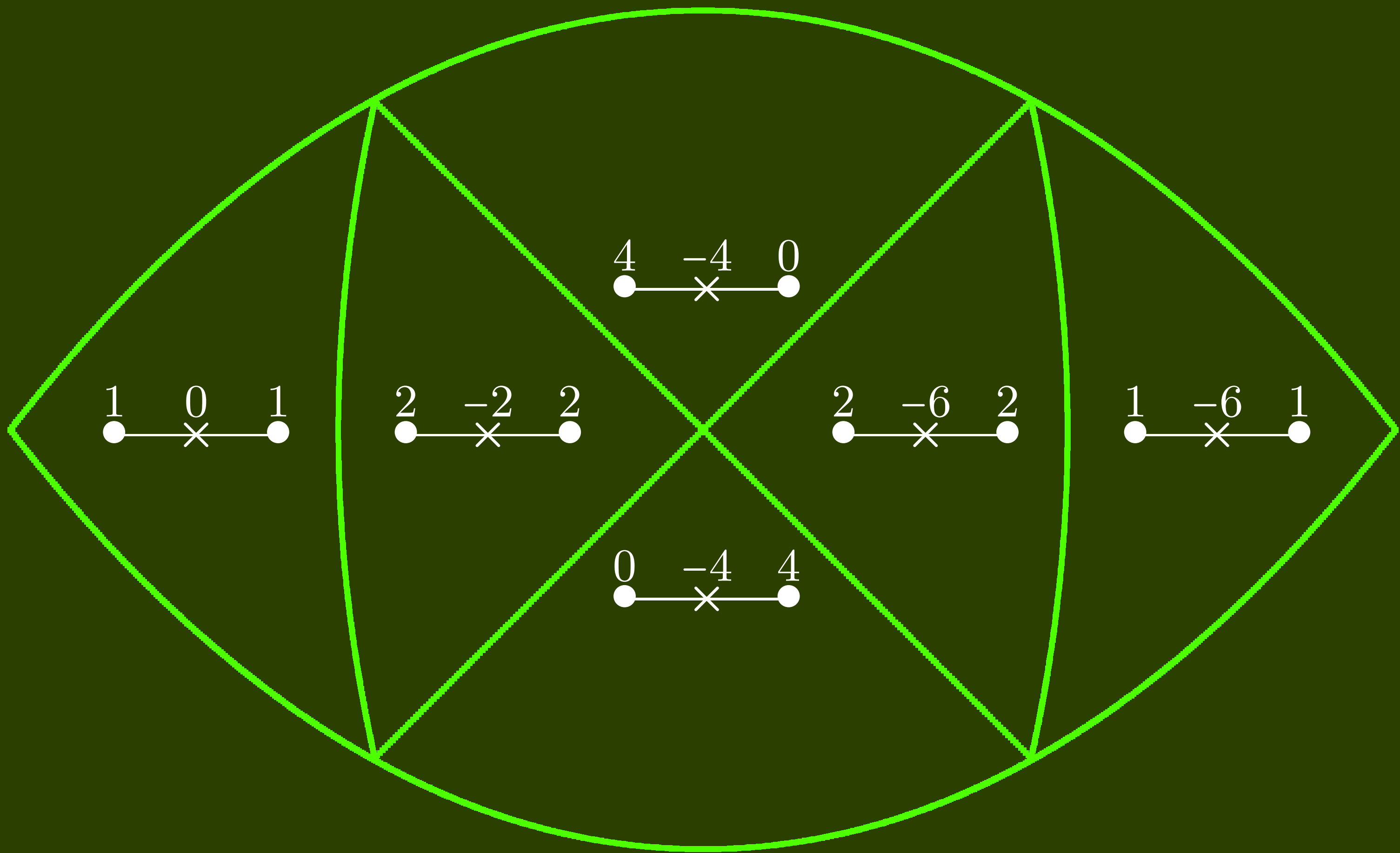


Can view as
a lune on a sphere!

The countries are
A3 Weyl chambers!

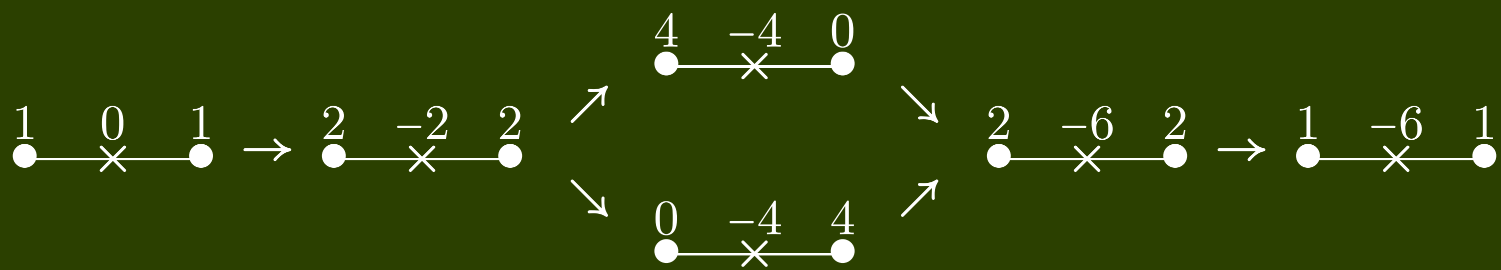
Conformal deformation complex

Recall tangent bundle $= S' \otimes S =$ 1 0 0 $\otimes$ 0 0 1 $=$ 1 0 1



Affine action of the Weyl group of A3 $(\lambda \mapsto w(\lambda + \rho) - \rho)$

Conformal deformation complex cont'd



Curved version (deformation sequence)

Vector Fields X^a

Conformal Perturbations $g_{ab} + \epsilon h_{ab}$

Self-dual Weyl Curvature

Anti-self-dual Weyl Curvature

Conformal Killing operator

$X^a \mapsto h_{ab} \equiv \nabla_{(a} X_{b)}$ − trace

Second order!

Bianchi/Bach

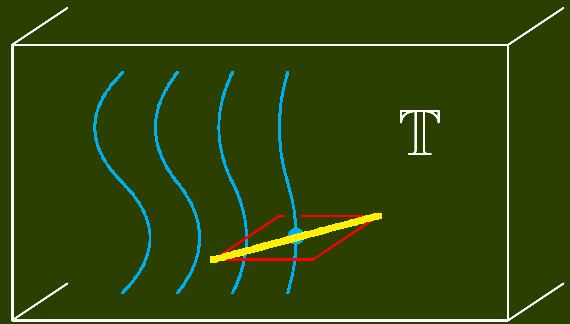
Twistor construction

Recall $\mathrm{Spin}(2,2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \rightsquigarrow$ Spin bundles

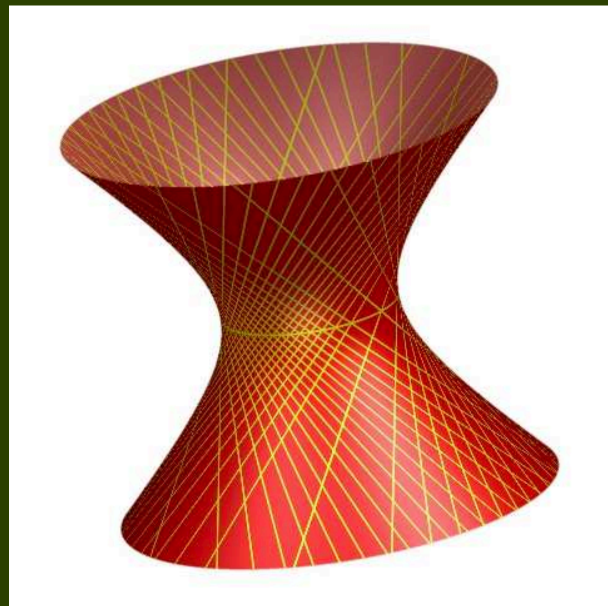
$TM = S' \otimes S$ null vectors = simple vectors

Segre $\mathbb{RP}_1 \times \mathbb{RP}_1 \hookrightarrow \mathbb{RP}_3$ nonsingular quadric as image

$$\begin{array}{c}\mathbb{P}(S') \\ \downarrow \\ M\end{array}$$



horizontal subspace



The blue lines and yellow planes are conformally invariant

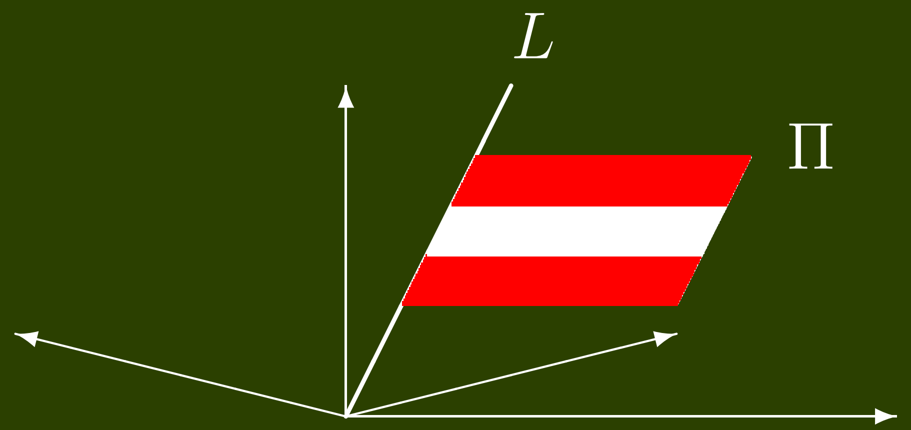
Five variables geometry

There are two! Let $\mathbb{T}$ be a 5-dimensional manifold.

- $\ell \oplus D \subset T\mathbb{T}$ s.t. $[D, D] \subseteq \ell \oplus D$ and $[\ell \oplus D, \ell \oplus D] = T\mathbb{T}$
- $D \subset T\mathbb{T}$ s.t. $[D, [D, D]] = T\mathbb{T}$ $\quad$ $(2, 3, 5)$ geometry

Flat models are generalised flag manifolds

- $\mathbb{F}_{1,2}(\mathbb{R}^4) = \{L \subset \Pi \subset \mathbb{R}^4$ s.t. $\dim L = 1, \ \dim \Pi = 2\}$



- $G/P = G_2/P =$ 

Twistor correspondence

$$\mathrm{SL}(4,\mathbb{R})\Big/\left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\}$$

$$\|$$

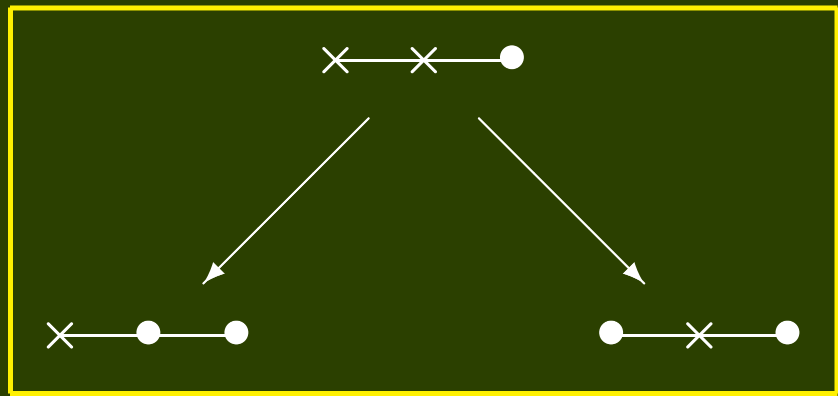
$$\mathbb{F}_{1,2}(\mathbb{R}^4)$$

$$\swarrow \qquad \searrow$$

$$\mathbb{RP}_3 \qquad\qquad \mathrm{Gr}_2(\mathbb{R}^4)$$

$$\| \qquad\qquad \|$$

$$\mathrm{SL}(4,\mathbb{R})\Big/\left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}\right\} \qquad \mathrm{SL}(4,\mathbb{R})\Big/\left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\}$$



An-Nurowski construction

$$\left.\begin{array}{rl} \text{Riemannian} & (\Sigma_1, g_1) \\ \text{Surfaces} & (\Sigma_2, g_2) \end{array}\right\} \rightsquigarrow M \equiv (\Sigma_1 \times \Sigma_2, g_1 \times -g_2)$$

$$\Downarrow$$

$$\mathbb{T}(M)$$

$$\|$$

Configuration space

of Σ_1 rolling on Σ_2

- Twistor structure $\ell \oplus D$ (five variables)
- Suppose $[D, D] = \ell \oplus D$ (generic)
- Forget ℓ but retain D (cinq variables)
- EG (Bryant) spheres of radii 1 and 3: G_2-flat
- (An-Nurowski) new G_2-flat examples rolling on a plane!

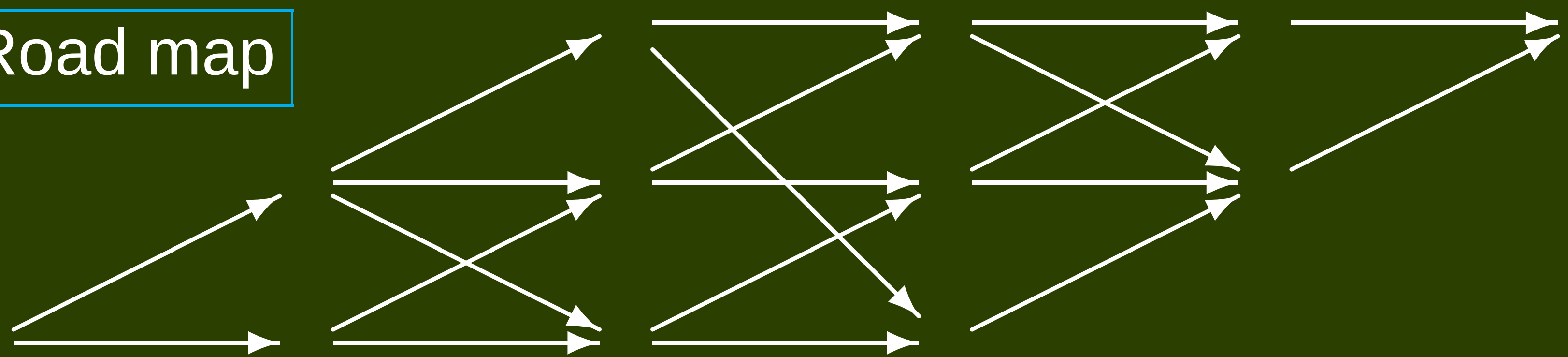
Differential complexes in five variables

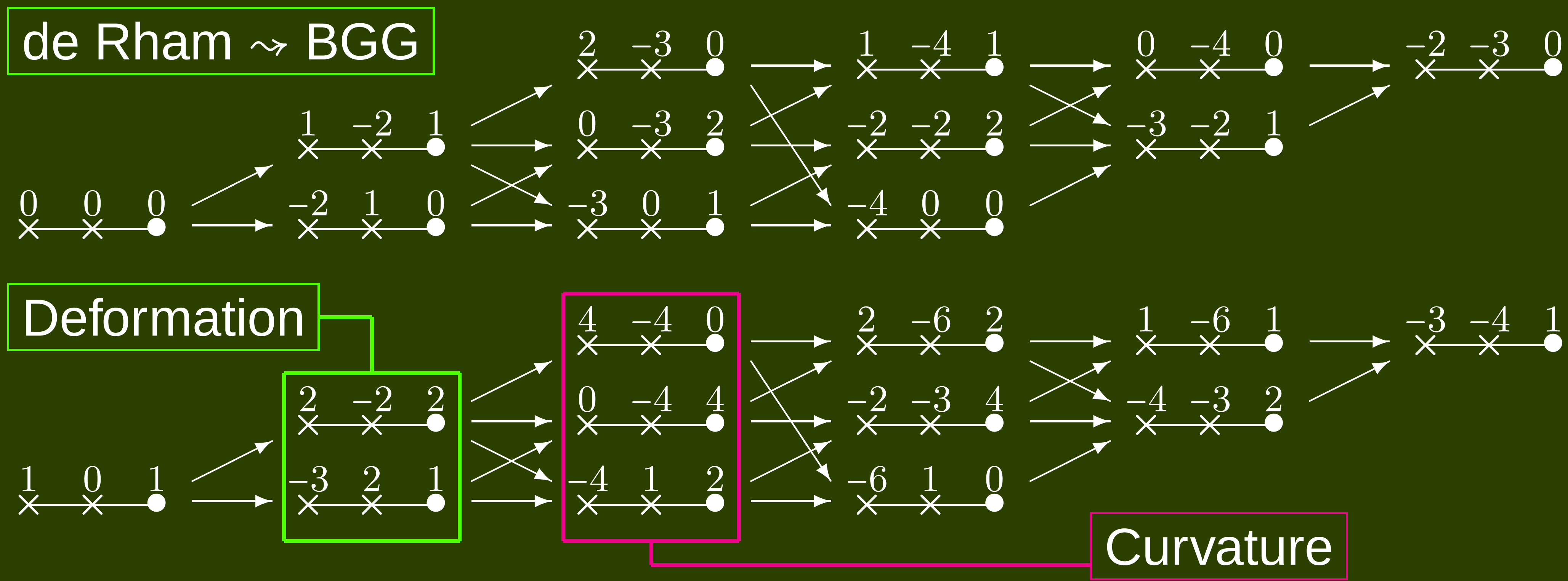
Parabolic subgroups of $\mathrm{SL}(4,\mathbb{R})$
$\stackrel{1:1}{\longleftrightarrow}$ lunes compatible with the A3 tiling of the sphere

$$\mathbb{F}_{1,2}(\mathbb{R}^4) = \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet \leftrightarrow \text{a hemisphere}$$

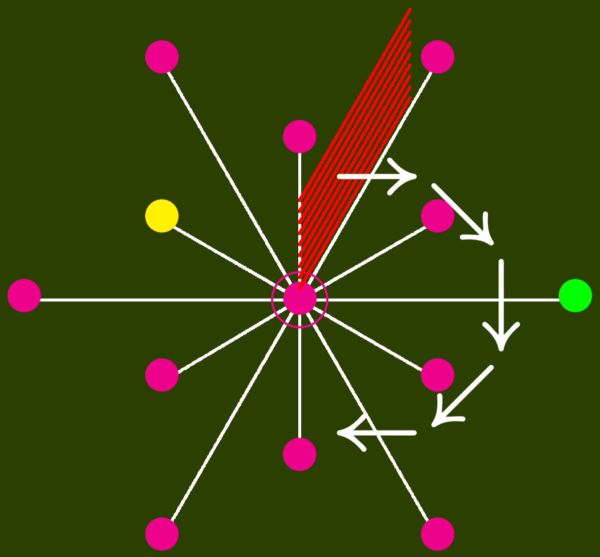
Differential complexes cont'd

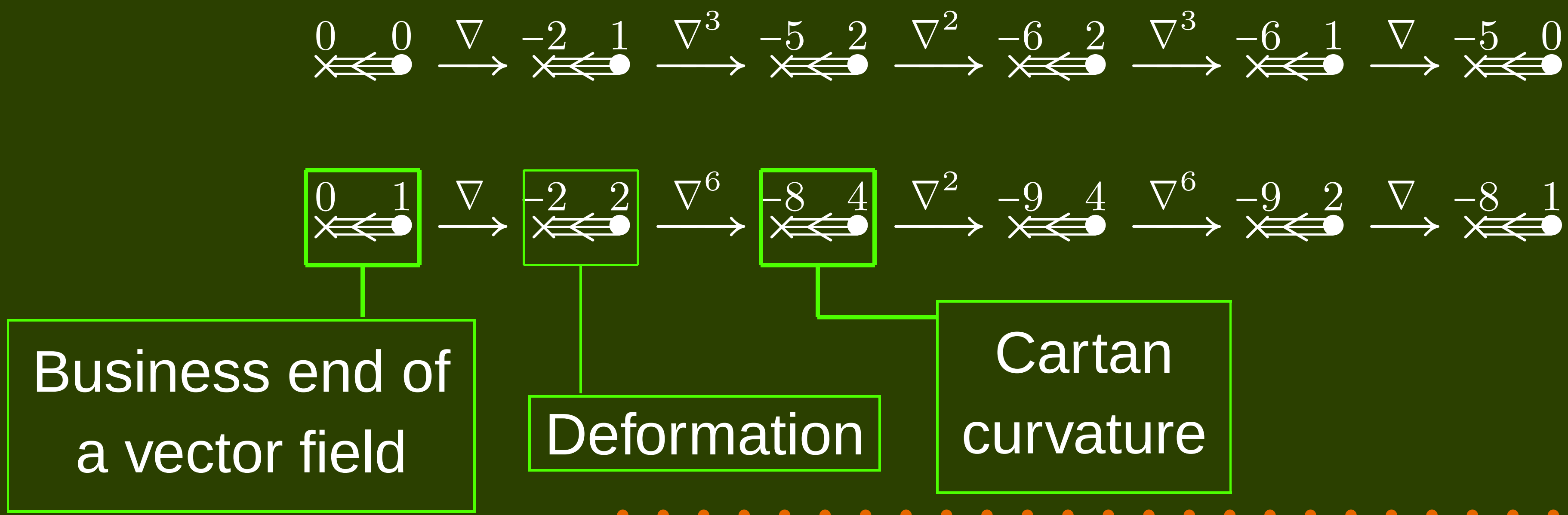
Road map

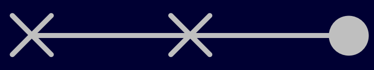
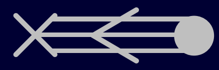


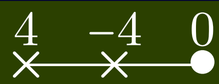


G2 road map

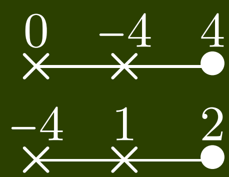



0 0 ∇ −2 1 ∇3 −5 2 ∇2 −6 2 ∇3 −6 1 ∇ −5 0
0 1 ∇ −2 2 ∇6 −8 4 ∇2 −9 4 ∇6 −9 2 ∇ −8 1
Business end of a vector field
Deformation
Cartan curvature

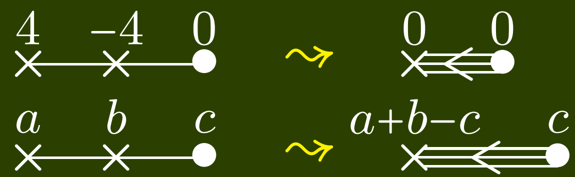
From  to 

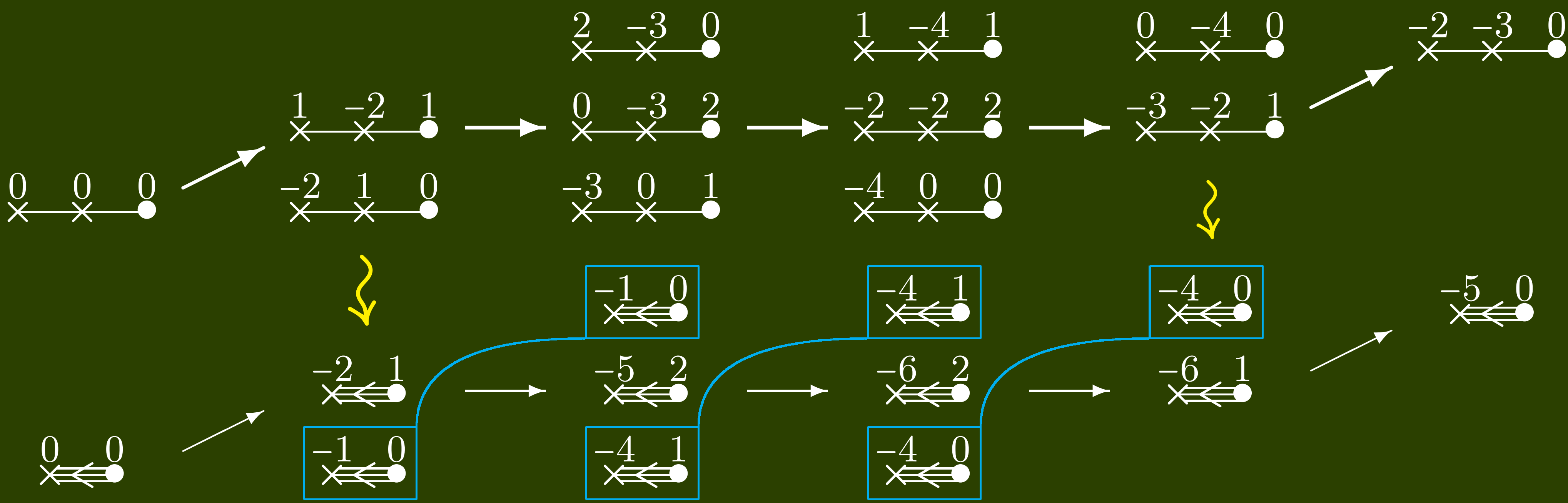
Recall 

Obstruction to integrability of D

Curvature 

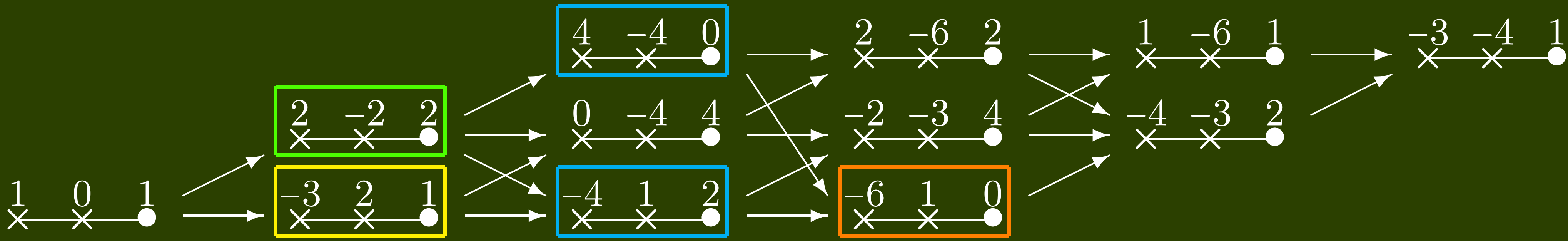
Vanishes on $\mathbb{T}($  $)$

$[D, D] = \ell \oplus D$ triggers collapse 

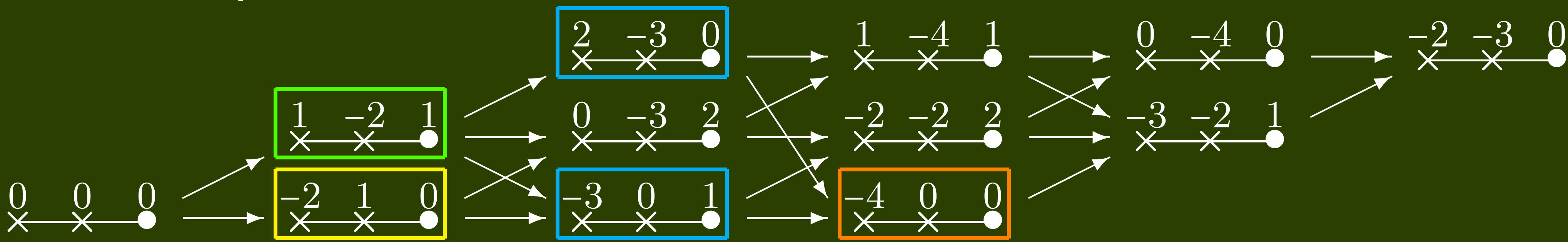


Theorem

Flat model



Compare



$\Longrightarrow$ Translate

There are (2,3,5) geometries that do not arise from a neutral signature conformal structure via the An-Nurowski twistor construction.

Easy algorithms!

THE END

THANK YOU