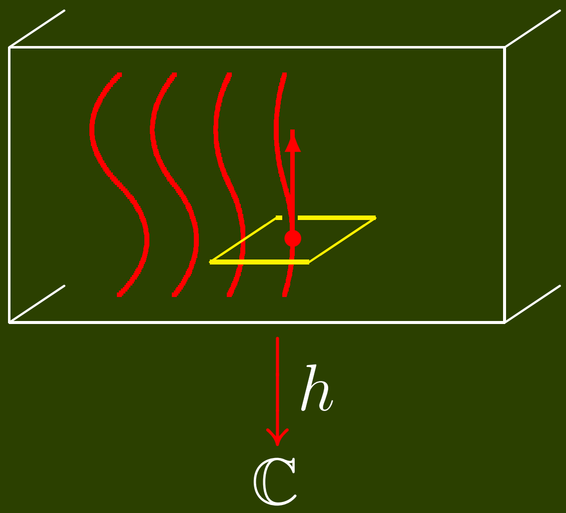
Some twistor constructions revisited

Michael Eastwood

Australian National University

Conformal foliations in $\mathbb{R}^3$

U = unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$.



U is (transversally) conformal $\Leftrightarrow \mathcal{L}_U$ preserves the conformal metric orthogonal to its leaves

isothermal coördinates

$\rightsquigarrow$

$$h = f + ig \qquad \langle \nabla f, \nabla g \rangle = 0$$
$$\|\nabla f\| = \|\nabla g\|$$

conjugate functions

Paul Baird & ME arXiv:1011.4717

Conjugate functions on $\mathbb{R}^3$

$$f = f(q,r,s) \quad g = g(q,r,s) \quad \text{s.t.} \begin{cases} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{cases}$$

- $f = r \quad g = s$
- $f = q^2 - r^2 - s^2 \quad g = 2q\sqrt{r^2 + s^2}$
- $f = r\dfrac{q^2 + r^2 + s^2}{r^2 + s^2} \quad g = s\dfrac{q^2 + r^2 + s^2}{r^2 + s^2}$
- $$f = \frac{(1 - q^2 - r^2 - s^2)r + 2qs}{r^2 + s^2}$$
  $$g = \frac{(1 - q^2 - r^2 - s^2)s - 2qr}{r^2 + s^2}$$

$$\begin{array}{ccc} \mathbb{R}^3 \hookrightarrow & S^3 & \\ & \downarrow & \text{Hopf} \\ \mathbb{R}^2 \leftarrow & S^2 \setminus \{*\} & \end{array}$$

Almost Hermitian structures

NB: $J(p,q,r,s) : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies

- $J^2 = -\mathrm{Id}$
- $J \in \mathrm{SO}(4)$

$$\Longleftrightarrow J = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & o & -u \\ w & -v & u & 0 \end{bmatrix}$$

$u^2 + v^2 + w^2 = 1$, two-sphere

Consider $\mathbb{R}^3 = \{(p,q,r,s) \in \mathbb{R}^4 \,|\, p = 0\} \subset \mathbb{R}^4$

NB: $U \equiv \left(J \frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3} = \left(u\frac{\partial}{\partial q} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial s}\right)\Big|_{\mathbb{R}^3}$

unit vector field

also $\rightsquigarrow$ two-sphere

Sphere bundles

bundle of unit vectors

bundle of almost Hermitian structues

$$\begin{array}{ccc} Q_\circ & \subset & Z_\circ \\ \downarrow & & \tau \downarrow \\ \mathbb{R}^3 & \subset & \mathbb{R}^4 \end{array}$$

section
$\updownarrow$
unit vector field

section
$\updownarrow$
almost Hermitian structure

Hermitian structures

Lemma

J is integrable $\Longrightarrow U \equiv \left(J\frac{\partial}{\partial p}\right)\Big|_{\mathbb{R}^3}$ is conformal

Conversely??

NB: J integrable $\Longrightarrow J$ real-analytic

Question: U conformal $\Longrightarrow U$ real-analytic??

Answer: NO!

However: U real-analytic and conformal
$\Longrightarrow U$ extends uniquely to an integrable J.

WHY?

Twistor geometry

bundle of almost Hermitian structues

$$\begin{array}{ccccl} Q_\circ & \subset & Z_\circ & = & \mathbb{CP}_3 \setminus \{z_3 = z_4 = 0\} \ni [z_1, z_2, z_3, z_4] \\ \downarrow & & \tau\downarrow & & \tau\downarrow \qquad\qquad\qquad \mapsto \\ \mathbb{R}^3 & \subset & \mathbb{R}^4 & = & \mathbb{C}^2 \ni \begin{pmatrix} p+iq \\ r+is \end{pmatrix} = \frac{1}{|z_3|^2+|z_4|^2}\begin{pmatrix} z_2\bar{z}_3 + z_4\bar{z}_1 \\ z_1\bar{z}_3 - z_4\bar{z}_2 \end{pmatrix} \end{array}$$

compactify

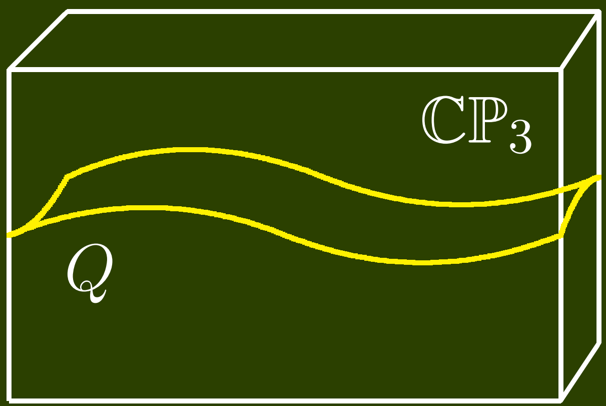
$$\begin{array}{c} \mathbb{CP}_3 \\ \tau\downarrow \\ S^4 \end{array}$$

twistor fibration (cf. Hopf)

Twistor geometry cont'd

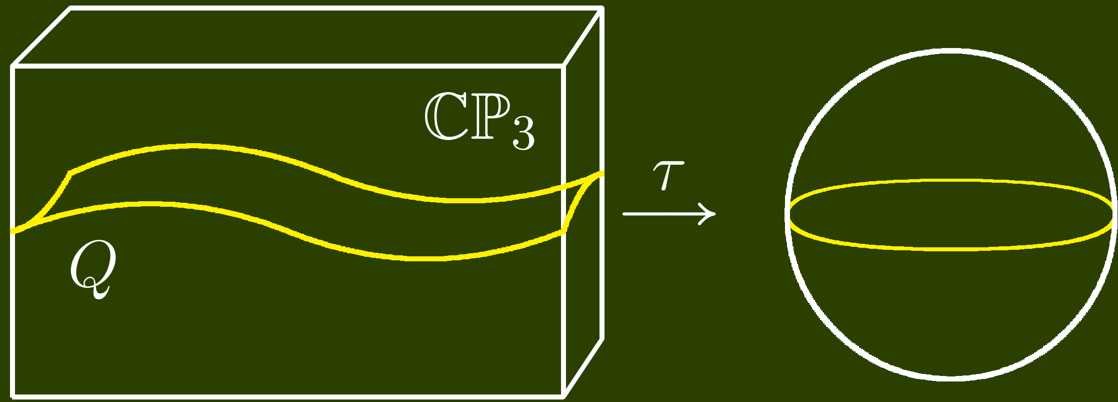
$$\begin{array}{ccc} Q_\circ & \subset & Z_\circ \\ \downarrow & & \downarrow \\ \mathbb{R}^3 & \subset & \mathbb{R}^4 \end{array} \quad \overset{\text{compactify}}{\hookrightarrow} \quad \begin{array}{ccc} Q & \subset & \mathbb{CP}_3 \\ \downarrow & & \tau \downarrow \\ S^3 & \subset & S^4 \end{array}$$

$$\begin{aligned} Q &= \{[z] \in \mathbb{CP}_3 \,|\, \Re(z_2\bar{z}_3 + z_4\bar{z}_1) = 0\} \\ &\cong \{[Z] \in \mathbb{CP}_3 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} \\ &\equiv \text{Levi-indefinite hyperquadric} \end{aligned}$$



Twistor results

$$\begin{array}{ccc} Q & \subset & \mathbb{CP}_3 \\ \downarrow & & \tau \downarrow \\ S^3 & \subset & S^4 \end{array}$$



Theorem A section $S^4 \supseteq^{\text{open}} \Omega \xrightarrow{J} \mathbb{CP}_3$ of τ defines an integrable Hermitian structure if and only if $\tilde{M} \equiv J(\Omega)$ is a complex submanifold.

Theorem A section $S^3 \supseteq^{\text{open}} \Omega \xrightarrow{U} Q$ of $\tau : Q \to S^3$ defines a conformal foliation if and only if $M \equiv U(\Omega)$ is a CR submanifold.

CR submanifolds and functions

$M \subset Q \subset \mathbb{CP}_3$ is a 'CR submanifold'?

It means: $TM \cap JTQ$ is preserved by J.

It does not mean: $M = \{f = 0\}$ where f is a

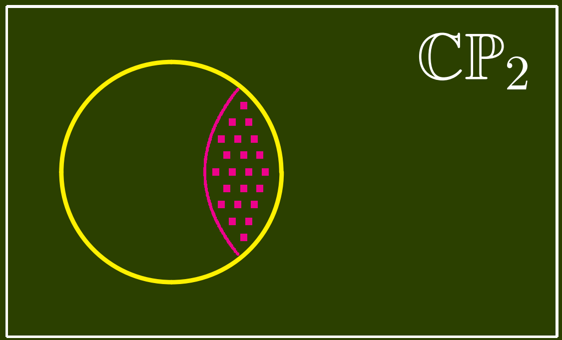
CR function: $(X + iJX)f = 0 \ \ \forall X \in \Gamma(TQ \cap JTQ)$.

Implicit function theorem
is false in the CR category

- CR functions on Q are real-analytic.
- conformal foliations on S^3 need not be.

CR functions

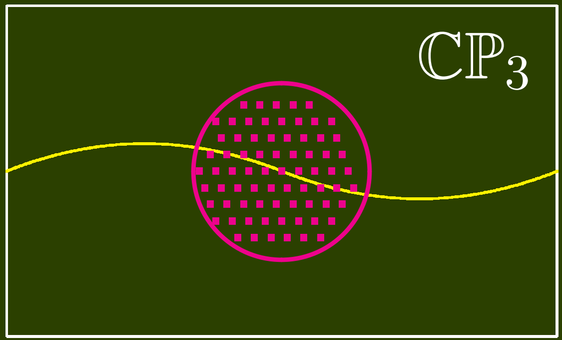
$\{[Z] \in \mathbb{CP}_2 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2\}$ = three-sphere



Theorem (H. Lewy 1956)

CR $\Rightarrow$ holomorphic extension

$\{[Z] \in \mathbb{CP}_3 \,|\, |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} = Q$



Corollary

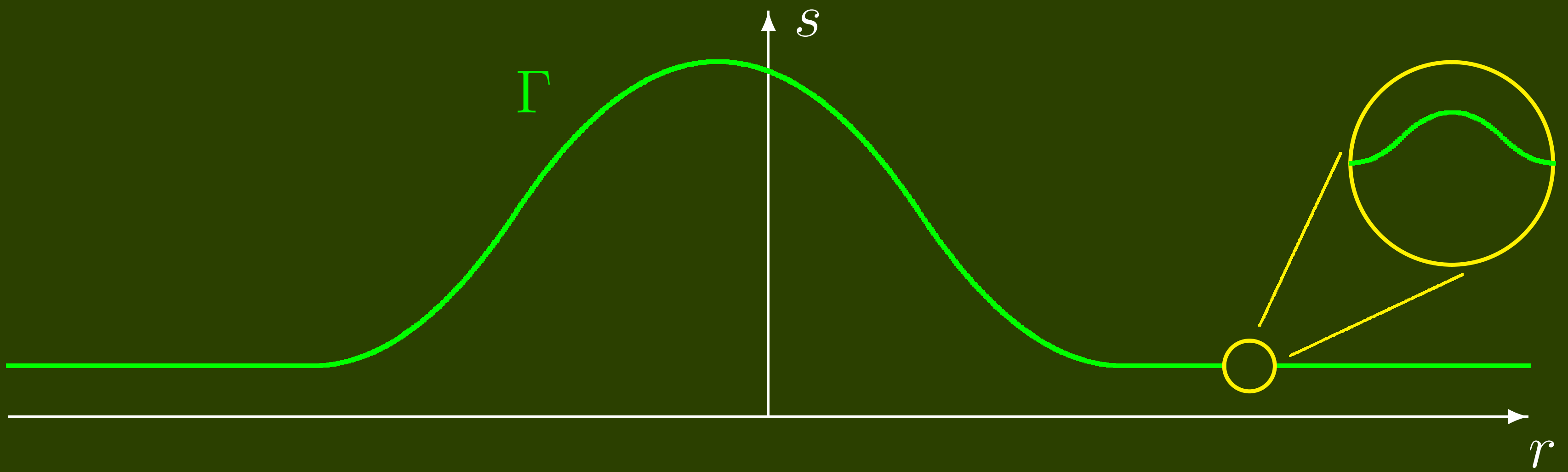
CR $\Rightarrow$ holomorphic extension

Hence, a CR function on Q is real-analytic!

Smooth conjugate functions

Eikonal equation: $\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial s}\right)^2 = 1$

Plenty of non-analytic solutions:



f = signed distance to Γ

$$\left.\begin{aligned} f(q,r,s) &= f(r,s) \\ g(q,r,s) &= q \end{aligned}\right\} \Rightarrow \begin{aligned} \langle \nabla f, \nabla g \rangle = 0 \\ \|\nabla f\| = \|\nabla g\| \end{aligned}$$

QED

Penrose transform

Recall the twistor fibration $\begin{array}{c}\mathbb{CP}_3\\ \tau\downarrow\\ S^4\end{array}$ For any $U^{\text{open}} \subseteq S^4$

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\simeq} \{\phi \in \Gamma(S^4, \mathcal{E}[-1]) \text{ s.t. } (-\Delta + \tfrac{1}{6}R)\phi = 0\}$$ Yamabe

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-3)) \xrightarrow{\simeq} \{\phi \in \Gamma(S^4, \mathcal{E}_{A'}[-1]) \text{ s.t. } \nabla^{AA'}\phi_{A'} = 0\}$$ Dirac

$$\mathcal{P} : \frac{H^1(\tau^{-1}(U), \Omega^1)}{d\,H^1(\tau^{-1}(U), \mathcal{O})} \xrightarrow{\simeq} \{\phi \in \Gamma(S^4, \mathcal{E}) \text{ s.t. } (\Delta^2 + \cdots)\phi = 0\}$$ Paneitz

$$\mathcal{P} : H^1(\tau^{-1}(U), \Theta) \xrightarrow{\simeq} \frac{\ker(\nabla^2 + \cdots) : \Gamma(U, \square\!\square_\circ) \to \Gamma(U, \boxplus_{\circ +})}{\operatorname{im} \nabla : \Gamma(U, \square) \to \Gamma(U, \square\!\square_\circ)}$$ $\dfrac{\text{Weyl}}{\text{Killing}_\circ}$

NB True for $U = S^4$! (Harmonic analysis via complex analysis)

Penrose transform for $\mathbb{CP}_2$

Another twistor fibration

$$\begin{array}{ccc} \mathbb{F}_{1,2}(\mathbb{C}^3) & \quad & L \subset P \\ \tau \downarrow & & \updownarrow \\ \mathbb{CP}_2 & & L^\perp \cap P \end{array}$$

NB

- $\mathbb{CP}_2$ with Fubini-Study metric (anti-self-dual)
- Penrose transform still valid (mutatis mutandis)
- Can compute $H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}(V))$ by Bott-Borel-Weil

EG

- $H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) = 0 \implies \mathbb{CP}_2$ is anti-self-dual rigid
- $\rightsquigarrow \mathbb{CP}_2$ is Blaschke rigid (cf. Tsukamoto 1981)

THE END

THANK YOU