The X-ray transform on complex projective space

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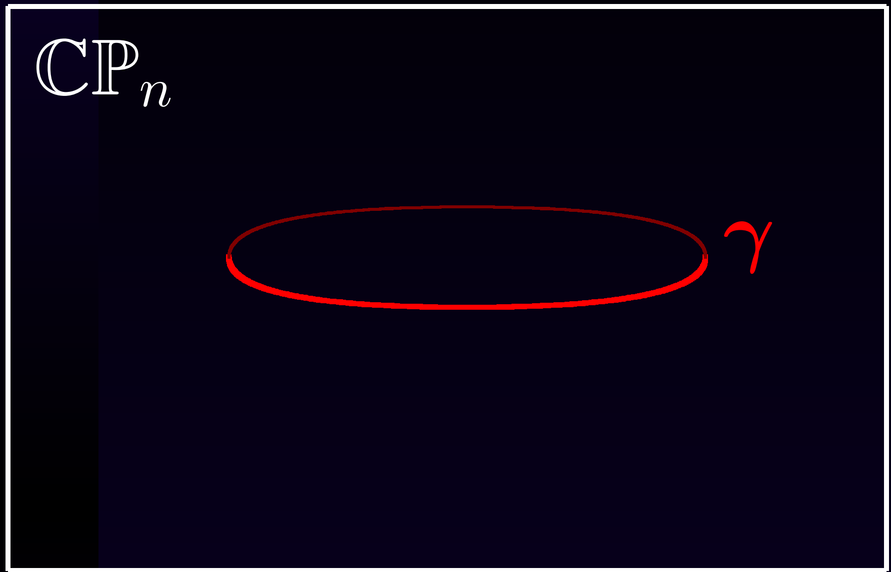
The X-ray transform



$\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$

Fubini-Study metric

f = smooth function on $\mathbb{CP}_n$

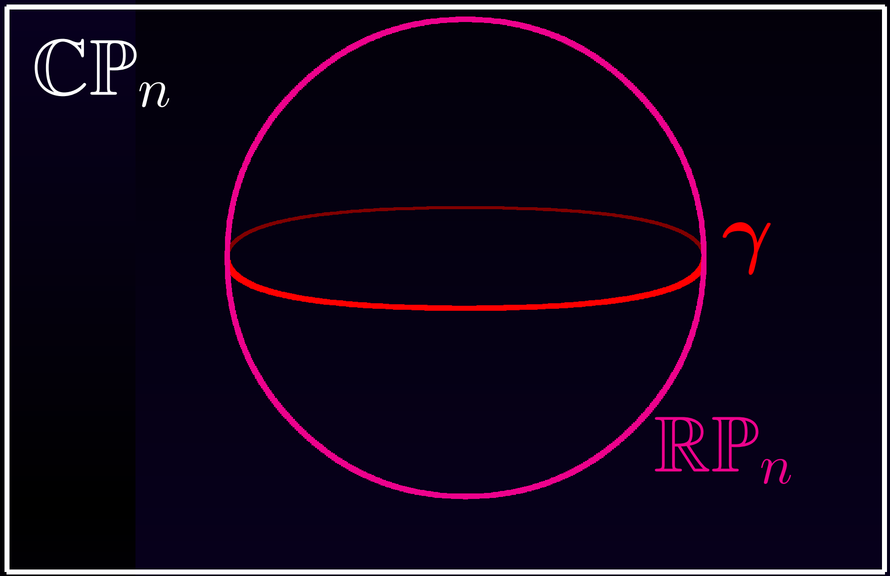
γ = geodesic

$$f \stackrel{\mathcal{X}}{\longmapsto} \phi(\gamma) = \oint_\gamma f$$

Questions

- Kernel of $\mathcal{X}$?
- What about $\omega \stackrel{\mathcal{X}}{\longmapsto} \phi(\gamma) = \oint_\gamma \omega$ for ω a 1-form?
- What about $\omega_{ab\cdots c}$ a symmetric tensor?

Method (with Hubert Goldschmidt)



$\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ induced by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is totally geodesic.

Translates by $\mathrm{SU}(n+1)$ too!

↑ ‘Model Embeddings’

- The X-ray transform on $\mathbb{RP}_n$ is well-understood.
- Restriction of tensors from $\mathbb{CP}_n$ to $\mathbb{RP}_n$ is OK.
- Suitable global techniques on $\mathbb{CP}_n$ are available,
- compatible with similar techniques on $\mathbb{RP}_n$.

Results

Observation:

- Suppose $\phi_{b\cdots c}$ is a smooth symmetric tensor on $\mathbb{CP}_n$.
- Let $\omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)}$ (Fubini-Study connection).
- Then $\mathcal{X}(\omega_{ab\cdots c}) = 0$.

Theorem (already known for $\mathbb{RP}_n$)

Conversely, $\mathcal{X}(\omega_{ab\cdots c}) = 0 \implies \omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)}$.

Example

For a 1-form ω on $\mathbb{CP}_n$, the following are equivalent.

- $\oint_\gamma \omega = 0 \;\; \forall$ geodesics γ.
- $\omega = d\phi$ for some smooth function ϕ.
- $d\omega = 0 \;\;$ (since $H^1(\mathbb{CP}_n, \mathbb{R}) = 0$).

Functions

Suppose

$$\oint_\gamma f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_\gamma f = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ for any model embedding. Hence (Funk 1913, Radon 1917),

$$f|_{\mathbb{RP}_n} = 0 \;\; \forall \text{ model embeddings } \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence $f = 0$, i.e. $\mathcal{X}$ is injective on functions

cf. Helgason, The Radon Transform, §2 Corollary 2.3

One-forms

Suppose

$$\oint_\gamma \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_\gamma \omega = 0 \; \forall$ geodesics $\gamma \hookrightarrow \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ for any model embedding. Hence (Michel 1978),

$$d\omega|_{\mathbb{RP}_n} = 0 \;\; \forall \text{ model embeddings } \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence $d\omega = \theta J$, for $J =$ the Kähler form on $\mathbb{CP}_n$.

Hence $d\omega = 0$ and so $\omega = d\phi$ for some ϕ.

Key points

Recall: model embeddings $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. Let $n \geq 2$.

- On $\mathbb{RP}_n$: $\oint_\gamma \omega = 0,\ \forall \gamma \iff \omega = d\phi$
- On $\mathbb{RP}_n$: $\omega = d\phi \iff d\omega = 0$
- On $\mathbb{CP}_n$: $\psi|_{\mathbb{RP}_n} = 0,\ \forall$ models $\iff \psi = \theta J$
- On $\mathbb{CP}_n$: $d\omega = \theta J \iff \omega = d\phi$

$\{$Model embeddings through $p \in \mathbb{CP}_n\} \leftrightarrow$
$\{$Lagrangian subspaces of $T_p\mathbb{CP}_n\} \ldots$
Helgason, Geometric Analysis ..., Exercise I.A.4(ii)

$d\omega = \theta J \Rightarrow 0 = d\theta \wedge J \Rightarrow d\theta = 0 \Rightarrow$
$\theta =$ constant $\Rightarrow \theta = 0 \Rightarrow d\omega = 0 \Rightarrow \omega = d\phi$

Symmetric two-tensors

- On $\mathbb{RP}_n$: $\oint_\gamma \omega_{ab} = 0,\ \forall\gamma \iff \omega_{ab} = \nabla_{(a}\phi_{b)}$

(Michel 1973) NB: $\phi_b \mapsto \nabla_{(a}\phi_{b)} =$ Killing operator

- On $\mathbb{RP}_n$: normalise $\boxed{R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}}$

Then, $\omega_{ab} = \nabla_{(a}\phi_{b)} \iff$

$$\nabla_{(a}\nabla_{c)}\omega_{bd} - \nabla_{(b}\nabla_{c)}\omega_{ad} - \nabla_{(a}\nabla_{d)}\omega_{bc} + \nabla_{(b}\nabla_{d)}\omega_{ac} + g_{ac}\omega_{bd} - g_{bc}\omega_{ad} - g_{ad}\omega_{bc} + g_{bd}\omega_{ac} = 0$$

In other words,

$$\boxed{\omega_{ab} = \nabla_{(a}\phi_{b)} \iff \pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right) = 0,}$$

where $\pi : \square\!\square \otimes \square\!\square \to \begin{smallmatrix}\square\!\square\\ \square\!\square\end{smallmatrix}$ $\leftrightsquigarrow$ canonical projection.

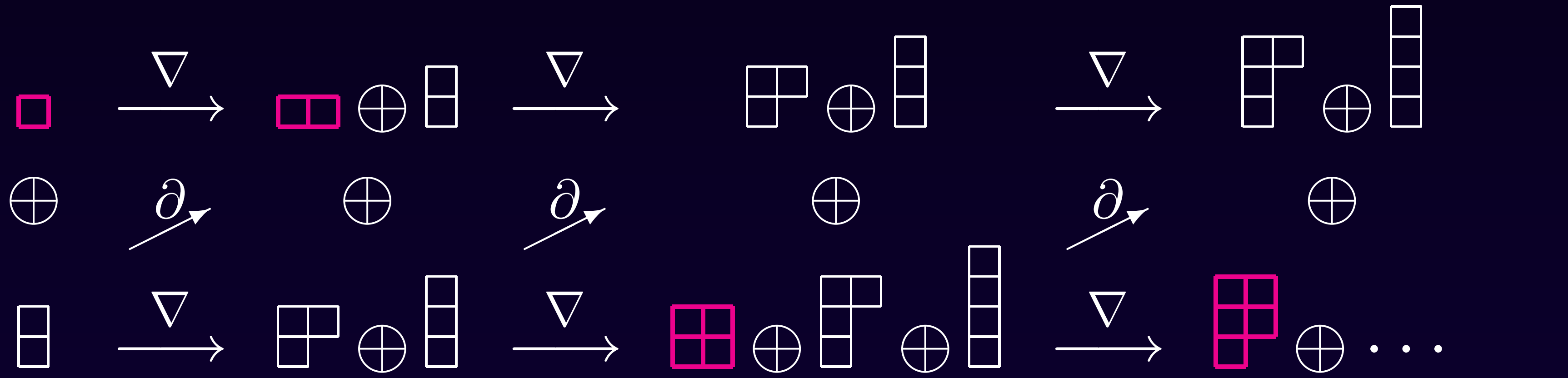
Proof of $\omega_{ab}=\nabla_{(a}\phi_{b)} \iff \pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd}+g_{ab}\omega_{cd}\right)=0$

Consider the connection on $\mathbb{V}\equiv\Lambda^1\oplus\Lambda^2$ given by

$$\begin{bmatrix}\sigma_b\\ \mu_{bc}\end{bmatrix}\stackrel{\nabla_a}{\longmapsto}\begin{bmatrix}\nabla_a\sigma_b-\mu_{ab}\\ \nabla_a\mu_{bc}+g_{ab}\sigma_c-g_{ac}\sigma_b\end{bmatrix}.$$

It is flat! Unravel the exactness of

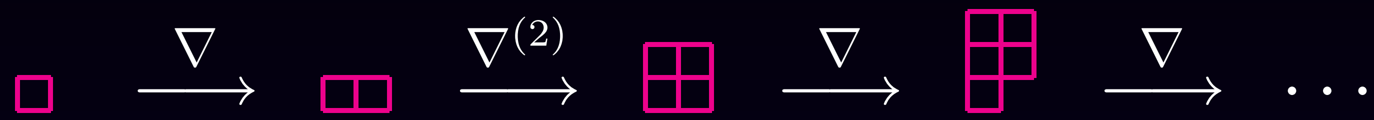
$$\Gamma(\mathbb{RP}_n,\mathbb{V})\xrightarrow{\nabla}\Gamma(\mathbb{RP}_n,\Lambda^1\otimes\mathbb{V})\xrightarrow{\nabla}\Gamma(\mathbb{RP}_n,\Lambda^2\otimes\mathbb{V})\xrightarrow{\nabla}$$



Lie algebra cohomology! (Kostant 1961)

BGG resolutions

The locally exact complexes



$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots$$

de Rham

are Bernstein-Gelfand-Gelfand resolutions.

$$\mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} * & * \cdots * \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix} \right\} = G/P,$$

where G is semisimple and P is parabolic.
As a Riemannian manifold, $\mathbb{RP}_n$ is projectively flat.

Some linear algebra

- On $\mathbb{CP}_n$: suppose $\psi_{ab} = \psi_{[ab]}$. Then

$$\psi_{ab}\big|_{\mathbb{RP}_n} = 0 \ \forall \text{ models} \iff \psi_{ab} = \theta J_{ab}.$$

Suppose $\psi_{abcd} = \psi_{[ab][cd]}$ and $\psi_{[abc]d} = 0$. Then

$$\begin{aligned}\psi_{abcd}\big|_{\mathbb{RP}_n} = 0 \ \forall \text{ models} \iff \\ \psi_{abcd} = \Psi_{ac}J_{bd} - \Psi_{bc}J_{ad} - \Psi_{ad}J_{bc} + \Psi_{bd}J_{ac} \\ + 2\Psi_{ab}J_{cd} + 2\Psi_{cd}J_{ab}, \quad \text{where } \Psi_{ab} = \Psi_{[ab]}.\end{aligned}$$

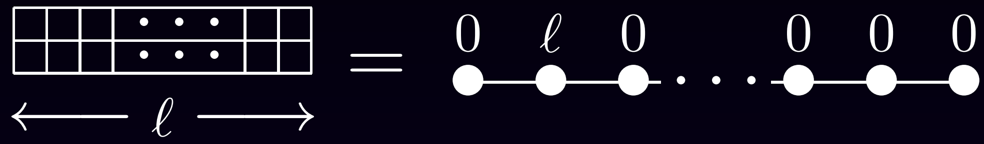
Rephrase: denote J-trace-free part of ψ by $\psi^\perp$

$$\psi_{ab} = \psi^\perp_{ab} + \theta J_{ab} \qquad \psi_{abcd} = \psi^\perp_{abcd} + \cdots$$

Then $\psi\big|_{\mathbb{RP}_n} = 0 \ \forall$ models $\iff \psi^\perp = 0$.

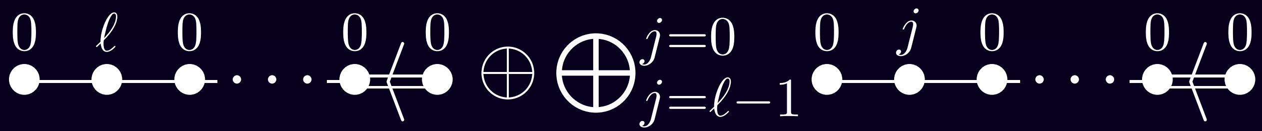
Symplectic branching

$\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$



$(2n - 1$ nodes).

Branching to $\mathrm{Sp}(2n, \mathbb{R})$ gives



$(n$ nodes).

Location of J-trace-free-part $\psi^{\perp}$.

For these tensors, Schur's Lemma $\ldots \Rightarrow$

$$\boxed{\psi|_{\mathbb{RP}_n} = 0 \ \forall \text{ models} \iff \psi^{\perp} = 0}.$$

Range of the Killing operator

- On $\mathbb{CP}_n$: $\omega_{ab} = \nabla_{(a}\phi_{b)} \iff$????????????

Recall

- On $\mathbb{RP}_n$: $\omega_a = \nabla_a\phi \iff \nabla_{[a}\phi_{b]} = 0$
- On $\mathbb{CP}_n$: $\omega_a = \nabla_a\phi \iff (\nabla_{[a}\phi_{b]})^\perp = 0$
- On $\mathbb{RP}_n$: $\omega_{ab} = \nabla_{(a}\phi_{b)} \iff$

$$\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right) = 0$$

- On $\mathbb{CP}_n$: $\omega_{ab} = \nabla_{(a}\phi_{b)} \iff$

$$\left(\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right)\right)^\perp = 0$$

Proof of $\Longrightarrow$:

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Symplectic geometry

Rumin-Seshadri complex

$$\boxed{\Lambda^0} \xrightarrow{d} \boxed{\Lambda^1} \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp \xrightarrow{d_\perp} \cdots \xrightarrow{d_\perp} \Lambda^n_\perp$$

$$\downarrow d_\perp^{(2)}$$

$$\Lambda^0 \xleftarrow{d_\perp} \Lambda^1 \xleftarrow{d_\perp} \Lambda^2_\perp \xleftarrow{d_\perp} \Lambda^3_\perp \xleftarrow{d_\perp} \cdots \xleftarrow{d_\perp} \Lambda^n_\perp$$

$\boxed{}$ local cohomology $= \mathbb{R}$

- $\Gamma(\mathbb{CP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp)$

is exact.

Symplectic geometry cont'd

Suppose ∇ is a connection on $\mathbb{V}$ such that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma = J_{ab}\Phi\Sigma \qquad \Phi \in \mathrm{End}\mathbb{V}.$$

Then we can couple the Rumin-Seshadri complex

$$\mathbb{V} \xrightarrow{\nabla} \Lambda^1 \otimes \mathbb{V} \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes \mathbb{V} \longrightarrow \cdots$$

It's still a complex and†

$$\Gamma(\mathbb{CP}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \mathbb{V})$$

is exact.

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† under further mild conditions

Proof of $\omega_{ab} = \nabla_{(a}\phi_{b)} \iff \left(\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right)\right)^{\perp} = 0$

Consider the connection on $\mathbb{V} \equiv \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$

$$\left[\begin{array}{c} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b + J_{ab}\rho_c - J_{ac}\rho_b - J_{bc}\rho_a + J_{bc}J_a{}^d\sigma_d \\ \nabla_a\rho_b + J_a{}^d\mu_{bd} \end{array}\right]$$

It satisfies† $(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = J_{ab}\Phi\Sigma$.

Unravel the exactness of

$$\Gamma(\mathbb{CP}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \mathbb{V})$$

again using Lie algebra cohomology! $\not\leftrightarrow$ BGG!!

† and further mild conditions

Final result

$$\oint_\gamma \omega_{ab} = 0 \; \forall\gamma \iff \omega_{ab} = \nabla_{(a}\phi_{b)}$$ (Tsukamoto 1981)

- On $\mathbb{RP}_n$: $\oint_\gamma \omega_{ab} = 0, \; \forall\gamma \Leftrightarrow \omega_{ab} = \nabla_{(a}\phi_{b)}$
- On $\mathbb{RP}_n$:
  $\omega_{ab} = \nabla_{(a}\phi_{b)} \Leftrightarrow \pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right) = 0$
- On $\mathbb{CP}_n$: $\psi_{abcd}|_{\mathbb{RP}_n} = 0, \; \forall$ models $\Leftrightarrow \psi^\perp_{abcd} = 0$
- On $\mathbb{CP}_n$:
  $\left(\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right)\right)^\perp = 0 \Leftrightarrow \omega_{ab} = \nabla_{(a}\phi_{b)}$

↑ Proof extends to general symmetric tensors ↑

THANK YOU

THE END