Conformal geometry and twistor theory

Higher symmetries of the Laplacian

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References

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- ME and Thomas Leistner, *Higher symmetries of the square of the Laplacian*, IMA Volumes 144, Springer Verlag 2007, pp. 319–338.

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A simple question on $\mathbb{R}^n$, $n \geq 3$

Question: Which linear differential operators preserve harmonic functions? Answer on $\mathbb{R}^3$:–

Zeroth order $f \mapsto$ constant $\times\, f$ 1

First order

$$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3 \qquad 3$$

$$x_1\nabla_2 - x_2\nabla_1 \quad \text{\&c.} \qquad 3$$

$$x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3 \boxed{+1/2} \qquad 1$$

$$({x_1}^2 - {x_2}^2 - {x_3}^2)\nabla_1 + 2x_1x_2\nabla_2 + 2x_1x_3\nabla_3 + x_1 \qquad 3$$

&c.

Dimensions . . . . $\boxed{10}$

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1$$

Lie Algebra $= \mathfrak{so}(4,1) =$ $\boxed{\text{conformal algebra}}$ $\leftarrow$ NB!

Second order

Boyer-Kalnins-Miller (1976)

Extras: $\propto$ Laplacian $(f \mapsto h\Delta f$ for any smooth $h)$
plus a 35-dim^{ℓ} family of new ones!

$$\{\mathcal{D}_1, \mathcal{D}_2\} \equiv \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1$$

$\bigodot^2 \mathfrak{so}(4,1) =?$ $\quad$ $\dim = 10 \times 11/2 = 55$

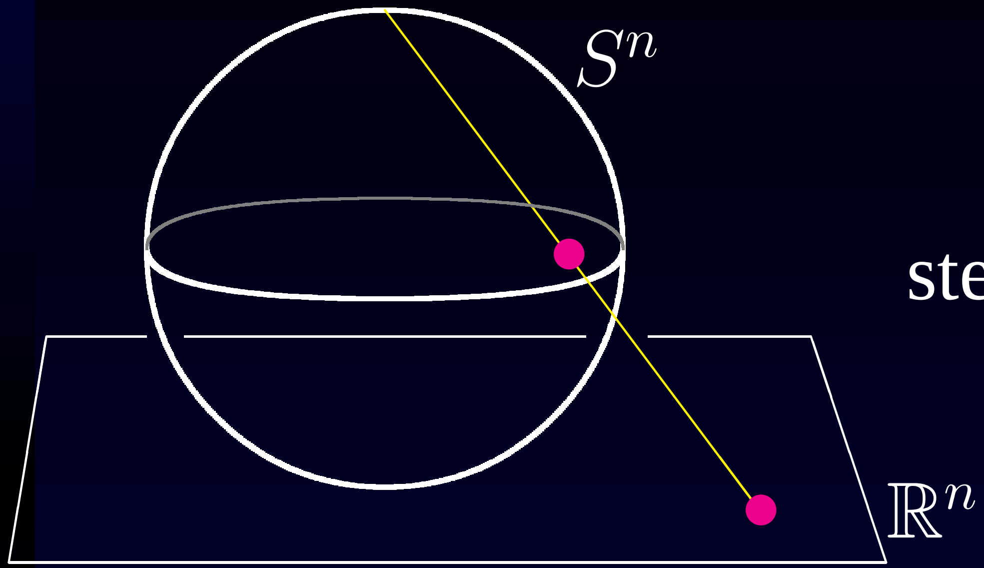
$$\yng(1,1) \odot \yng(1,1) = \yng(2,2)_{\circ} \oplus \yng(2)_{\circ} \oplus \mathbb{R} \oplus \yng(1,1,1,1)$$

$$55 = 35 + 14 + 1 + 5$$

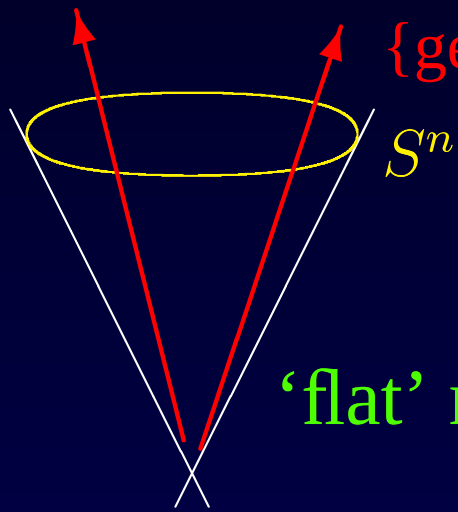
Separation of variables (Bôcher, Bateman, ...).

Third order...?

Conformal geometry



stereographic projection



'flat' model

Action of $\mathrm{SO}(n+1,1)$ on S^n
by conformal transformations

Conformal Laplacian **Dirac 1935**

$$r \equiv {x_1}^2 + \cdots {x_n}^2 + {x_{n+1}}^2 - {x_{n+2}}^2$$

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2} + \frac{\partial^2}{\partial {x_{n+1}}^2} - \frac{\partial^2}{\partial {x_{n+2}}^2}$$

f on null cone $\subset \mathbb{R}^{n+2}$ homogeneous of degree $w \rightsquigarrow$

- ambiently extend to $\tilde{f}$ of degree w
- freedom $\tilde{f} \mapsto \tilde{f} + rg$ for g of degree $w - 2$
- calculate: $\tilde{\Delta}(rg) = r\tilde{\Delta}g + 2(n + 2w - 2)g$

$w = 1 - n/2 \Rightarrow f \mapsto (\tilde{\Delta}\tilde{f})|_{r=0}$ is invariantly defined.

On $\mathbb{R}^n$ it's Δ

On S^n it's $\Delta - \frac{n-2}{4(n-1)}R$

AdS/CFT

Fefferman-Graham‘ambient’ metric

Symmetries of Δ

$\mathcal{D}$ a symmetry $\iff$ $\Delta\mathcal{D} = \delta\Delta$ for some δ.

trivial example: $\mathcal{D} = \mathcal{P}\Delta$ for any $\mathcal{P}$

equivalence: $\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$

$\mathbb{R}^n \rightsquigarrow \mathcal{A}_n \equiv$ algebra of symmetries

under composition up to equivalence

Write $\mathcal{D} = V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_d +$ lower order terms

symbol

normalise w.l.g. to be trace-free

Theorems

- $\mathcal{D}$ a symmetry $\Rightarrow$ trace-free part of $\nabla^{(a}V^{bc\cdots d)} = 0$
  - Easy
- On $\mathbb{R}^n$, such a conformal Killing tensor $V^{bc\cdots d} \rightsquigarrow \mathcal{D}_V$
  - Not So Easy

$\mathcal{D}_V$ is a canonically associated symmetry of the form

$$\mathcal{D}_V = V^{bc\cdots d}\nabla_b\nabla_c\cdots\nabla_d + \text{lower order terms.}$$

- E.g. First order

$$\mathcal{D}_V f = V^a\nabla_a f + \tfrac{n-2}{2n}(\nabla_a V^a)f$$

- E.g. Second order

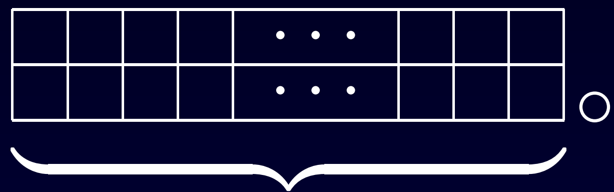
$$\mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f + \tfrac{n}{n+2}(\nabla_a V^{ab})\nabla_b f + \tfrac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b V^{ab})f$$

Ingredients of proof

- We can solve the conformal Killing tensor equation

$$\nabla^{(a}V^{bc\cdots d)} = g^{(ab}\lambda^{c\cdots d)}$$

on $\mathbb{R}^n$ by prolongation and/or BGG machinery:–



w.r.t. $\mathfrak{so}(n+1,1)$.

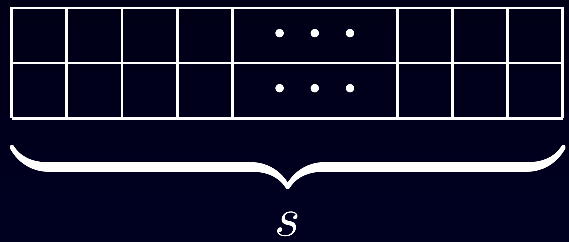
# of columns = # of indices on $V^{bc\cdots d}$

$$\begin{aligned} \text{E.g. } V^b &= s^b + m^{bc}x_c + \lambda x^b + r^c x_c x^b - \tfrac{1}{2}x^c x_c r^b \\ &= \text{translation} + \text{rotation} + \text{dilation} + \text{inversion.} \end{aligned}$$

- Use 'ambient' methods to construct $\mathcal{D}_V$.

Corollary

As a vector space

$$\mathcal{A}_n = \bigoplus_{s=0}^{\infty}$$  $$\circ$$

Question: What about the algebra structure?

Cf.: let $\mathfrak{g}$ be a Lie algebra. As a vector space

$$\mathfrak{U}(\mathfrak{g}) = \bigoplus_{s=0}^{\infty} \bigodot^{s} \mathfrak{g}$$

but the algebra structure is opaque viewed this way.

The algebra structure

$$\mathfrak{U}(\mathfrak{g}) = \frac{\bigotimes \mathfrak{g}}{(X \otimes Y - Y \otimes X - [X, Y])}$$

$$= \frac{\bigotimes \mathfrak{g}}{(X \otimes Y - X \odot Y - \frac{1}{2}[X, Y])}$$

Theorem

$$\mathcal{A}_n = \frac{\bigotimes \mathfrak{so}(n+1,1)}{(X \otimes Y - X \circledcirc Y - \frac{1}{2}[X, Y] + \frac{n-2}{4n(n+1)}\langle X, Y \rangle)}$$

Cartan Lie Killing

Equivalently,

$$\mathcal{A}_n = \mathfrak{U}(\mathfrak{so}(n+1,1))/\text{Joseph Ideal}.$$

Proof of algebra structure

Calculate by ambient means that

$$\mathcal{D}_X\mathcal{D}_Y = \mathcal{D}_{X\circledcirc Y} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} - \tfrac{n-2}{4n(n+1)}\mathcal{D}_{\langle X,Y\rangle}$$

and use properties of Cartan product (due to Kostant).

Remark: simple Lie algebra $= \mathfrak{g} \neq \mathfrak{sl}(2,\mathbb{C}) \Rightarrow$

$$\dim \frac{\bigotimes \mathfrak{g}}{(X\otimes Y - X\circledcirc Y - \frac{1}{2}[X,Y] - \lambda\langle X,Y\rangle)} = \infty$$

for precisely one value of λ (Braverman and Joseph)

$\rightsquigarrow$ graded algebra $\bigoplus_{s=0}^{\infty} \circledcirc^s \mathfrak{g}$.

Curved analogues

For any vector field V^a,

$$f \mapsto V^a \nabla_a f + \tfrac{n-2}{2n}(\nabla_a V^a) f$$

is conformally invariant.

For any trace-free symmetric tensor field V^{ab},

$$\begin{aligned} f \;\mapsto\; & V^{ab}\nabla_a\nabla_b f + \tfrac{n}{n+2}(\nabla_a V^{ab})\nabla_b f \\ & + \tfrac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b V^{ab}) f \\ & \boxed{- \tfrac{n+2}{4(n+1)} R_{ab} V^{ab} f} = \boxed{\text{curvature correction terms}} \end{aligned}$$

is conformally invariant &c. &c.

Curved symmetries?

- V^a is a conformal Killing vector $\Rightarrow$

$$\mathcal{D}_V f \equiv V^a \nabla_a f + \tfrac{n-2}{2n}(\nabla_a V^a) f$$

is symmetry of the conformal Laplacian.

- V^{ab} is a conformal Killing tensor $\overset{?}{\Longrightarrow}$

$$\begin{aligned}\mathcal{D}_V f \equiv\ & V^{ab}\nabla_a\nabla_b f + \tfrac{n}{n+2}(\nabla_a V^{ab})\nabla_b f \\ & + \tfrac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b V^{ab}) f - \tfrac{n+2}{4(n+1)} R_{ab} V^{ab} f\end{aligned}$$

is a symmetry of the conformal Laplacian. Unknown!

Another operator

In even dimensions, there is the Dirac operator

$$D : \mathbb{S}^+ \to \mathbb{S}^-.$$

A symmetry of D is an operator $\mathcal{D} : \mathbb{S}^+ \to \mathbb{S}^+$ s.t.

$$\begin{array}{ccc} \mathbb{S}^+ & \xrightarrow{\quad D \quad} & \mathbb{S}^- \\ \mathcal{D}\downarrow & & \delta\downarrow \\ \mathbb{S}^+ & \xrightarrow{\quad D \quad} & \mathbb{S}^- \end{array}$$

commutes for some differential operator $\delta : \mathbb{S}^- \to \mathbb{S}^-$.

- The symbol of $\mathcal{D}$ satisfies a conformally invariant overdetermined system of equations.

✓ First order symmetries: Benn and Kress.

✓ Higher order symmetries in the flat case: E, Somberg, and Souček.

Yet another operator

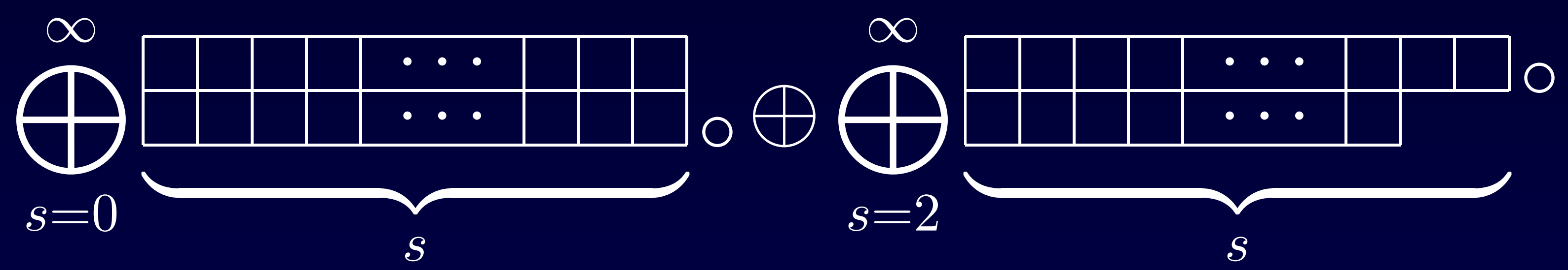
For the square of the Laplacian (E and Leistner) symmetry algebra =

$$\frac{\bigotimes \mathfrak{so}(n+1,1)}{\left(\begin{array}{l} X\otimes Y - X\circledcirc Y - X\bullet Y - \frac{1}{2}[X,Y] + \frac{(n-4)(n+4)}{4n(n+1)(n+2)}\langle X,Y\rangle \\ \text{and some fourth order elements}\end{array}\right)}$$

new

different

with graded counterpart



THANK YOU

THE END