Twistor theory and the harmonic hull

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Harmonic functions

The Laplacian on $\mathbb{R}^n$: $\Delta = \dfrac{\partial^2}{\partial {x_1}^2} + \dfrac{\partial^2}{\partial {x_2}^2} + \cdots + \dfrac{\partial^2}{\partial {x_n}^2}$

$\Delta u = 0 \implies u$ is real-analytic

$$\therefore \; u(x_1, x_2, \cdots, x_n) \rightsquigarrow \widehat{u}(z_1, z_2, \cdots, z_n)$$

More precisely,

$$\begin{array}{ccccccc} \mathbb{R}^n & \supseteq^{\text{open}} & U & \xrightarrow{u} & \mathbb{R} \text{ or } \mathbb{C} \\ \cap & & \cap & & \cap \\ \mathbb{C}^n & \supseteq^{\text{open}} & \widehat{U} & \xrightarrow{\widehat{u}} & \mathbb{C} \end{array}$$

FAQ: Depends on u? How big?

2 dimensions

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\underbrace{\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)}_{\text{Cauchy-Riemann}}$$

- $\mathbb{R}^2 \supseteq^{\text{open}} U \xrightarrow{u} \mathbb{C}$ harmonic
- U simply-connected

$\Longrightarrow$

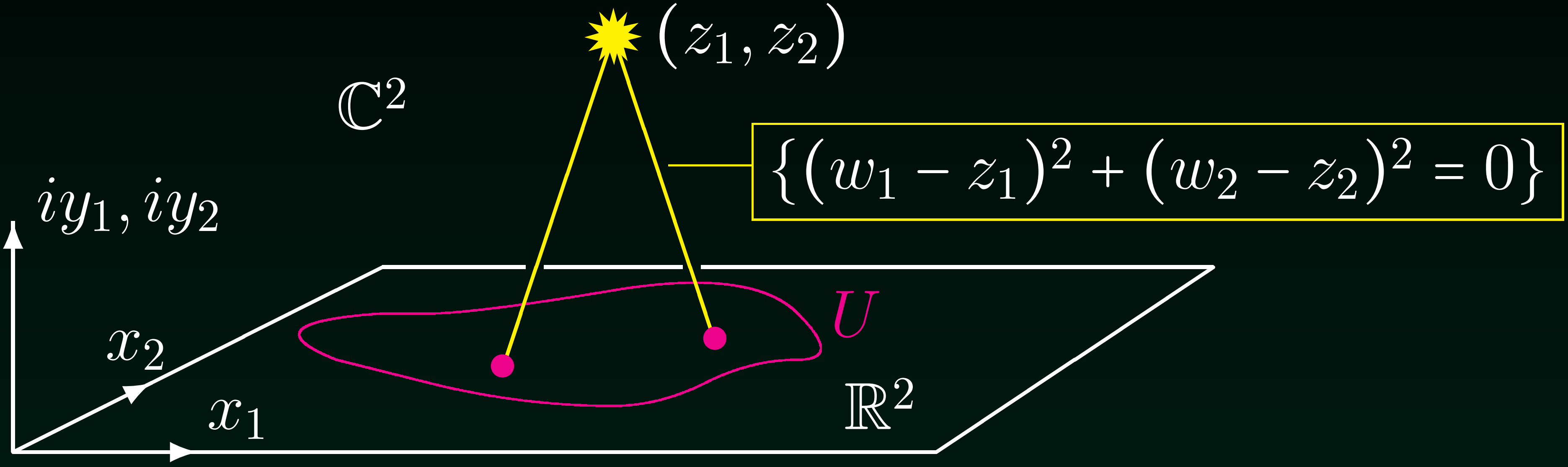
$$u(x_1, x_2) = \underbrace{f}_{\text{holomorphic}}(x_1 + ix_2) + \underbrace{g}_{\text{holomorphic}}(x_1 - ix_2)$$

$$\therefore\ \widehat{u}(z_1, z_2) = f(z_1 + iz_2) + g(z_1 - iz_2)$$

where it makes sense: $z_1 + iz_2 \in U$ and $z_1 - iz_2 \in \overline{U}$

2 dimensions cont'd

$z_1 + iz_2 \in U$ and $z_1 - iz_2 \in \overline{U}$ $\Longleftrightarrow$



$\therefore$ all u harmonic on U extend to $\widetilde{u}$ holomorphic on

$$\widetilde{U} \equiv \{z \in \mathbb{C}^2 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^2 \subset U\}$$

where $\mathcal{N}(z) = \{w \in \mathbb{C}^2 \text{ s.t. } (w - z)^2 = 0\}$ Null 'Cone'

2 dimensions concl'd

If $U \subseteq \mathbb{R}^2$ is simply-connected then

- all harmonic functions on U extend to $\widetilde{U} \subseteq \mathbb{C}^2$,
- $\widetilde{U}$ is connected (and simply-connected),
- $\widetilde{U}$ is maximal in this respect*.

$\widetilde{U}$ is called the harmonic hull of U.

$\star$ consider $\log(z - x)^2$ for $x \in \partial U$.

If $U \subset \mathbb{R}^2$ is multiply-connected, it does not have a harmonic hull: consider

$$\log(z - x)^2 \quad \text{for } x \in \mathbb{R}^2 \setminus U \text{ surrounded by } U.$$

4 dimensions

Bateman's formula

$$u(x) = \oint_\gamma f\,((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta)\, d\zeta$$

f: a holomorphic function of 3 variables

γ: a closed contour in the complex plane

Differentiation under the integral sign $\Longrightarrow$ $\boxed{\Delta u = 0}$

FAQ (circa 1904–1980)

- where is f defined?
- where is γ located?
- which u arise in this way?

answer by means of Penrose transform

Suspend disbelief!

4 dimensions cont'd

Naïve use of Bateman

$$u(x) = \oint_\gamma f\left((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta\right) d\zeta$$

$$\rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow \quad \rightsquigarrow$$

$$u(z) = \oint_\gamma f\left((z_1 + iz_2) + (iz_3 + z_4)\zeta, (iz_3 - z_4) + (z_1 - iz_2)\zeta, \zeta\right) d\zeta$$

Perhaps we should insist that

$f(z, \zeta)$ be defined wherever $f(x, \zeta)$ is defined i.e.

for fixed $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$,

$$L_z \equiv \{((z_1+iz_2)+(iz_3+z_4)\zeta, (iz_3-z_4)+(z_1-iz_2)\zeta, \zeta) \text{ s.t. } \zeta \in \mathbb{C}\}$$

$\cap$ WANT! $\swarrow$ open in $\mathbb{C}^3$

$$\{((x_1+ix_2)+(ix_3+x_4)\zeta, (ix_3-x_4)+(x_1-ix_2)\zeta, \zeta) \text{ s.t. } x \in U, \zeta \in \mathbb{C}\}$$

What we want

Fix $z \in \mathbb{C}^4$. For any $\zeta \in \mathbb{C}$, we want to be able to solve

$$\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix}\begin{bmatrix} 1 \\ \zeta \end{bmatrix} = \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix}\begin{bmatrix} 1 \\ \zeta \end{bmatrix}$$

for some $x = (x_1, x_2, x_3, x_4) \in U \subseteq^{\text{open}} \mathbb{R}^4$. Necessarily,

$$\det\left(\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix} - \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix}\right) = 0$$

i.e. $\boxed{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2 + (x_4 - z_4)^2 = 0}$

(also if $\zeta = \infty$). This is also sufficient: we require

$\boxed{\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \quad \text{where } \mathcal{N}(z) = \text{null cone based at } z}$

4 dimensions concl'd

For any $U^{\text{open, connected}} \subseteq \mathbb{R}^4$, let

$$\widetilde{U} \equiv \{z \in \mathbb{C}^4 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^4 \subset U\}^{\text{connected}}$$

where $\mathcal{N}(z) = \{w \in \mathbb{C}^4 \text{ s.t. } (w - z)^2 = 0\}$ Null Cone

- All harmonic functions on U extend to $\widetilde{U}$,
- $\widetilde{U}$ is maximal $\star$ (harmonic hull).

$\star$ Consider $1/(z - x)^2$ for $x \in \partial U$.

NB $n = 4$ is better than $n = 2$ if Bateman OK.
?????????????

The twistor fibration

Looking at Bateman's formula, consider

$$\begin{array}{ccc} \zeta \in \mathbb{C} & \hookrightarrow & \mathbb{C}^3 \\ \downarrow\!\!\!\!\!\!\top & & \Uparrow \\ \multicolumn{3}{c}{((x_1+ix_2)+(ix_3+x_4)\zeta, (ix_3-x_4)+(x_1-ix_2)\zeta, \zeta)} \end{array}$$

write L_x for the range, where $x \in \mathbb{R}^4$

Compactify! $\mathbb{CP}_1 \hookrightarrow \mathbb{CP}_3$ (still write range as L_x)

Foliation of $\mathbb{CP}_3 \setminus \{[*, *, 0, 0]\} \hookrightarrow \mathbb{CP}_3$

$$\begin{array}{ccccc} & \mathbb{CP}_3 \setminus \{[*,*,0,0]\} & \hookrightarrow & \mathbb{CP}_3 & \\ & \downarrow & & \downarrow \tau & \text{submersion} \\ L_x = \tau^{-1}(x) & \mathbb{R}^4 & \overset{\hookrightarrow}{\longleftarrow} & S^4 & \end{array}$$

stereographic projection

cf. Hopf

The Penrose transform

Theorem For any $U^{\text{open}} \subseteq \mathbb{R}^4$

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\cong} \{u : U \to \mathbb{C} \text{ s.t. } \Delta u = 0\}.$$

Interprets Bateman's formula and answers FAQ!

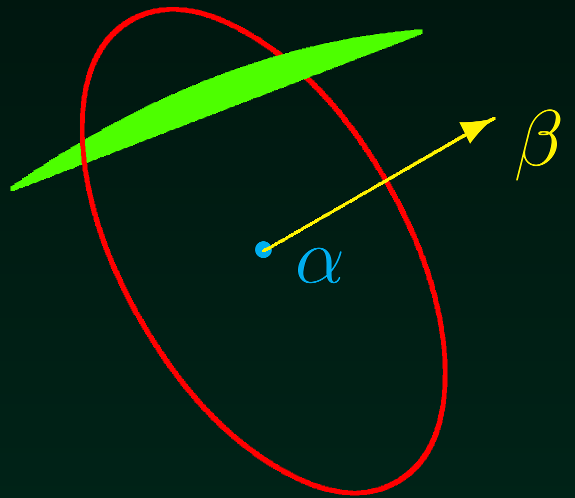
- $\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \iff L_z \subset \tau^{-1}(U)$
- $\widetilde{U}$ = harmonic hull, as suspected
- For $U^{\text{open}} \subseteq S^4$, $\quad \Delta \rightsquigarrow \Delta - \frac{1}{6}R \quad$ (Yamabe)
- OK in higher even dimensions (M.K. Murray).

Geometry of the harmonic hull

For $z = \alpha + i\beta \in \mathbb{C}^n$

$$\begin{aligned} \mathcal{N}(z) \cap \mathbb{R}^n &= \{x \text{ s.t. } (x-z)^2 = 0\} \\ &= \{x \text{ s.t. } |x-\alpha|^2 = |\beta|^2 \ \& \ (x-\alpha).\beta = 0\} \end{aligned}$$

NB $\mathcal{N}(z) \cap \mathbb{R}^n = \mathcal{N}(\overline{z}) \cap \mathbb{R}^n$



Odd dimensions

$U^{\text{open,connected}} \subseteq \mathbb{R}^{\text{odd}}$ need not have a harmonic hull!

Example $\quad \mathbb{R}^3 \setminus \{0\}$ $\quad$ Reasons

Recall $\mathbb{C}^3 \ni z \rightsquigarrow \mathcal{N}(z) \cap \mathbb{R}^3 \equiv \text{Circle}(z) \rightsquigarrow \text{Disc}(z)$

- OK extension to $\{z \text{ s.t. Disc}(z) \not\ni 0\}$
  Aronszajn, Creese, and Lipkin
- $f(x) \rightsquigarrow \frac{1}{\sqrt{1-2r.x+|r|^2|x|^2}} f\left(\frac{x-r|x|^2}{1-2r.x+|r|^2|x|^2}\right)$ harmonic ✓
- Hence should extend to $\mathbb{C}^3 \setminus \{{z_1}^2 + {z_2}^2 + {z_3}^2 = 0\}$
- But $1/\sqrt{{z_1}^2 + {z_2}^2 + {z_3}^2}$ branches!!

Cf. Huygens, Kirchkoff, et cetera

Further Reading

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THANK YOU