Differential complexes for the Grushin distributions

Michael Eastwood

[Ovidiu Calin Der-Chen Chang
Jan Slovák Vladimír Souček]

Australian National University

Grushin ‘distributions’

Standard coördinates (x, y) on $\mathbb{R}^2$ k^{th} Grushin fields

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv x^k \frac{\partial}{\partial y}$$

bracket generating

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

Cf $\frac{\partial}{\partial x} + ix\frac{\partial}{\partial y}$ Mizohata

Coördinates (x, y, t) on $\mathbb{R}^3$

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial y}$$

Martinet (Cf Darboux)

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + (ix - y)\frac{\partial}{\partial t}$$

Lewy (Cauchy-Riemann-Heisenberg)

Zeroth Grushin $\rightsquigarrow$ de Rham complex

Standard coördinates (x, y) on $\mathbb{R}^2$

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left.\begin{array}{rcl} Xf & = & g \\ Yf & = & h \end{array}\right\} \Rightarrow Yg - Xh = 0$$

locally the converse is true

On any smooth 2-manifold

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \to 0,$$

the de Rham complex.

The Rumin complex

Coördinates (x, y, t) on $\mathbb{R}^3$

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv \frac{\partial}{\partial t} + x\frac{\partial}{\partial y} \qquad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left.\begin{array}{rcl} Xf &=& g \\ Yf &=& h \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} XYg - X^2h + Zg &=& 0 \\ YXh - Y^2g - Zh &=& 0 \end{array}\right.$$

locally the converse is true

On any contact 3-manifold

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{d_H^{(2)}} \Lambda^1_H \otimes L \xrightarrow{d_H} \Lambda^3 \to 0,$$

the Rumin complex.

Rumin cont'd

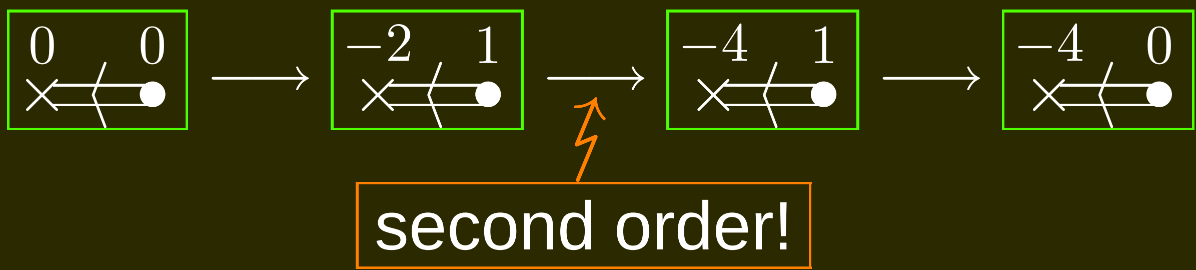
Contact 3-manifold: $H \subset TM$ such that $[H,H] = TM$.

bracket generating

$$0 \to L \to \Lambda^1 \to \Lambda^1_H \to 0$$

$$\begin{array}{ccccccccccccc} 0 & \to & \mathbb{R} & \to & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \to & 0 \\ & & & & \| & & \| & & \| & & \| & & \\ & & & & \boxed{\Lambda^0} & & \boxed{\Lambda^1_H} & & \Lambda^2_H & & & & \\ & & & & & & + & & + & & & & \\ & & & & & & L & \cong & \boxed{\Lambda^1_H \otimes L} & & \boxed{\Lambda^2_H \otimes L} & & \end{array}$$

Levi ($L \cong \Lambda^2_H$)



(2,3,5)-geometries

Traditional coördinates (x, y, p, q, z) on $\mathbb{R}^5$

$$X \equiv \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + q^2\frac{\partial}{\partial z} \qquad Y \equiv \frac{\partial}{\partial q}$$

generate a 2-plane distribution H in $\mathbb{R}^5$ and then

$$Z \equiv [X, Y] = -\frac{\partial}{\partial p} - 2q\frac{\partial}{\partial z}$$

and then

$$W \equiv [X, Z] = \frac{\partial}{\partial y} \qquad U \equiv [Y, Z] = -2\frac{\partial}{\partial z}$$

(and then everything else commutes).

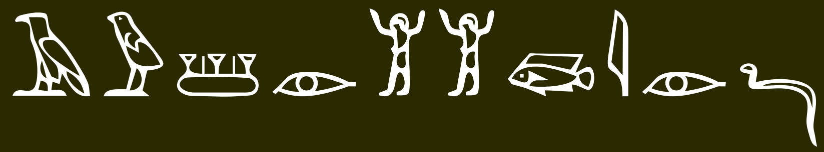
(2,3,5) cont'd

$H \subset TM$ $\qquad$ $\underbrace{H \subseteq [H,H] \subseteq [H,[H,H]] = TM}$

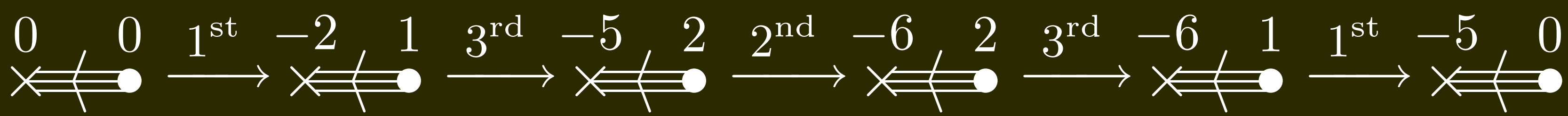
rank 2 $\quad$ rank 3 $\quad$ rank 5

bracket generating
and more besides

NB Frame $X, Y \in \Gamma(H)$. $\quad Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$0 \to \mathbb{R} \to \Lambda^0 \to \Lambda^1_H \to$??? 

For any $(2,3,5)$-geometry



BGG . . . $\quad$ R Bryant, ME, R Gover, K Neusser

First Grushin

Standard coördinates (x, y) on $\mathbb{R}^2$

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv x\frac{\partial}{\partial y} \qquad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left.\begin{array}{rcl} Xf &=& g \\ Yf &=& h \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} XYg - X^2h + Zg &=& 0 \\ YXh - Y^2g - Zh &=& 0 \end{array}\right.$$

conversely?

Locally (on $U^{\text{open}} \subseteq \mathbb{R}^2$ contractible)

$$\left.\begin{array}{rcl} Xf &=& g \\ Yf &=& h + C \end{array}\right\} \Leftrightarrow \left\{\begin{array}{rcl} XYg - X^2h + Zg &=& 0 \\ YXh - Y^2g - Zh &=& 0 \end{array}\right.$$

constant

First Grushin versus contact

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x\frac{\partial}{\partial y} \quad Z \equiv [X,Y] = \frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2$$

versus

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv \frac{\partial}{\partial t} + x\frac{\partial}{\partial y} \quad Z \equiv [X,Y] = \frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^3$$

NB $0 \to \mathcal{E}_2 \to \mathcal{E}_3 \xrightarrow{\partial/\partial t} \mathcal{E}_3 \to 0$ $\quad \partial/\partial t$ is a symmetry.

Symmetry reduction

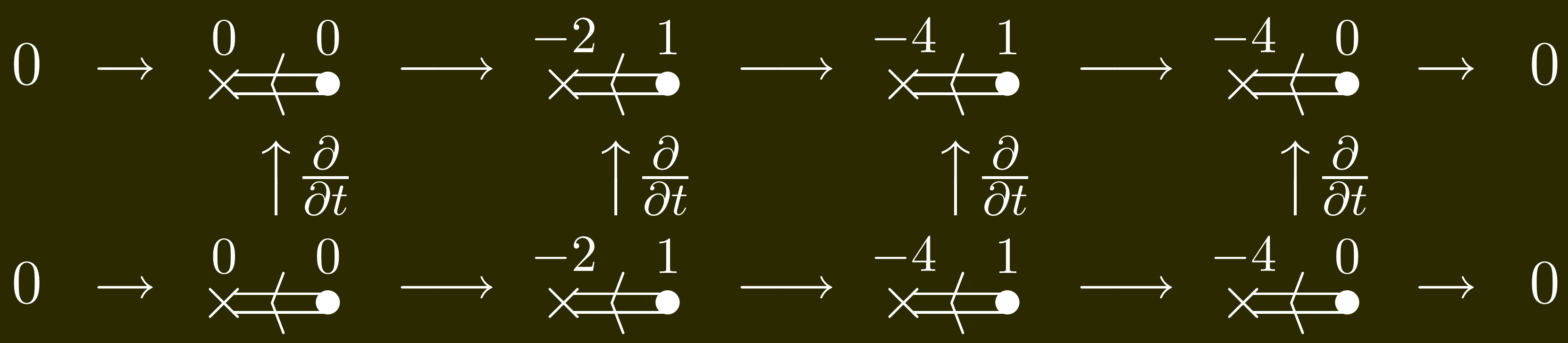


Diagram chasing $\rightsquigarrow$

First Grushin complex

$$X \equiv \frac{\partial}{\partial x} \qquad Y \equiv x\frac{\partial}{\partial y} \qquad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \begin{matrix}\mathcal{E}_2\\ \oplus \\ \mathcal{E}_2\end{matrix} \longrightarrow \begin{matrix}\mathcal{E}_2\\ \oplus \\ \mathcal{E}_2\end{matrix} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

$$f \longmapsto \begin{bmatrix} Xf \\ Yf \end{bmatrix} \quad \begin{bmatrix} g \\ h \end{bmatrix} \mapsto \begin{bmatrix} X^2h - (XY + Z)g \\ Y^2g - (YX - Z)h \end{bmatrix} \quad \begin{bmatrix} p \\ q \end{bmatrix} \longmapsto Xq + Yp$$

here

and here

else exact

Local cohomology is $\mathbb{R}$

O Calin, D-C Chang, ME

Second Grushin

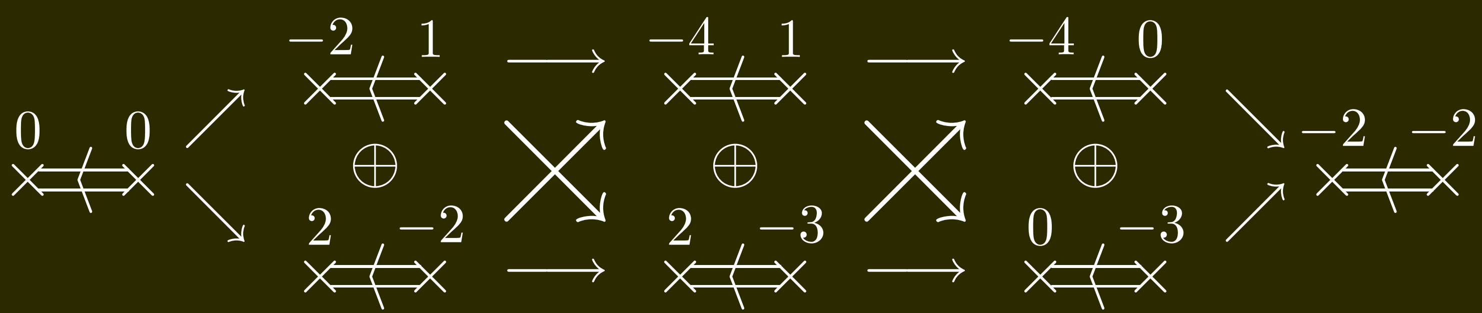
$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x^2\frac{\partial}{\partial y} \quad Z \equiv [X,Y] = 2x\frac{\partial}{\partial y} \quad W \equiv [X,Z] = 2\frac{\partial}{\partial y}$$

versus Engel on $\mathbb{R}^4$

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial s} + x\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial y}$$

$$Z \equiv [X,Y] = \frac{\partial}{\partial t} + 2x\frac{\partial}{\partial y} \qquad W \equiv [X,Z] = 2\frac{\partial}{\partial y}$$

BGG … R Bryant, ME, R Gover, K Neusser (B Doubrov)



Second Grushin complex

On $\mathbb{R}^4$

$$
\begin{array}{ccccccccc}
\Lambda^0 & \longrightarrow & \Lambda^1 & \longrightarrow & \Lambda^2 & \longrightarrow & \Lambda^3 & \longrightarrow & \Lambda^4 \\
\| & & \| & & \| & & \| & & \| \\
1\,\boxed{1} & & 2\,\boxed{2} & & 1 & & & & \\
 & & + & \nearrow & + & & & & \\
 & & 1 & & 2\,\boxed{1} & & 1 & & \\
 & & + & \nearrow & + & \nearrow & + & & \\
 & & 1 & & 2\,\boxed{1} & & 1 & & \\
 & & & & + & \nearrow & + & & \\
 & & & & 1 & & 2\,\boxed{2} & & 1\,\boxed{1}
\end{array}
$$

O Calin, D-C Chang, ME
arXiv:1207.5440

$\rightsquigarrow$

$$
0 \to \boxed{1} \xrightarrow{1^{\text{st}}} \boxed{2}
\begin{array}{c} \overset{2^{\text{nd}}}{\nearrow} \\ \overset{}{\searrow}_{3^{\text{rd}}} \end{array}
\begin{array}{c} \boxed{1} \\ \oplus \\ \boxed{1} \end{array}
\begin{array}{c} \overset{3^{\text{rd}}}{\searrow} \\ \nearrow_{2^{\text{nd}}} \end{array}
\boxed{2} \xrightarrow{1^{\text{st}}} \boxed{1} \to 0
$$

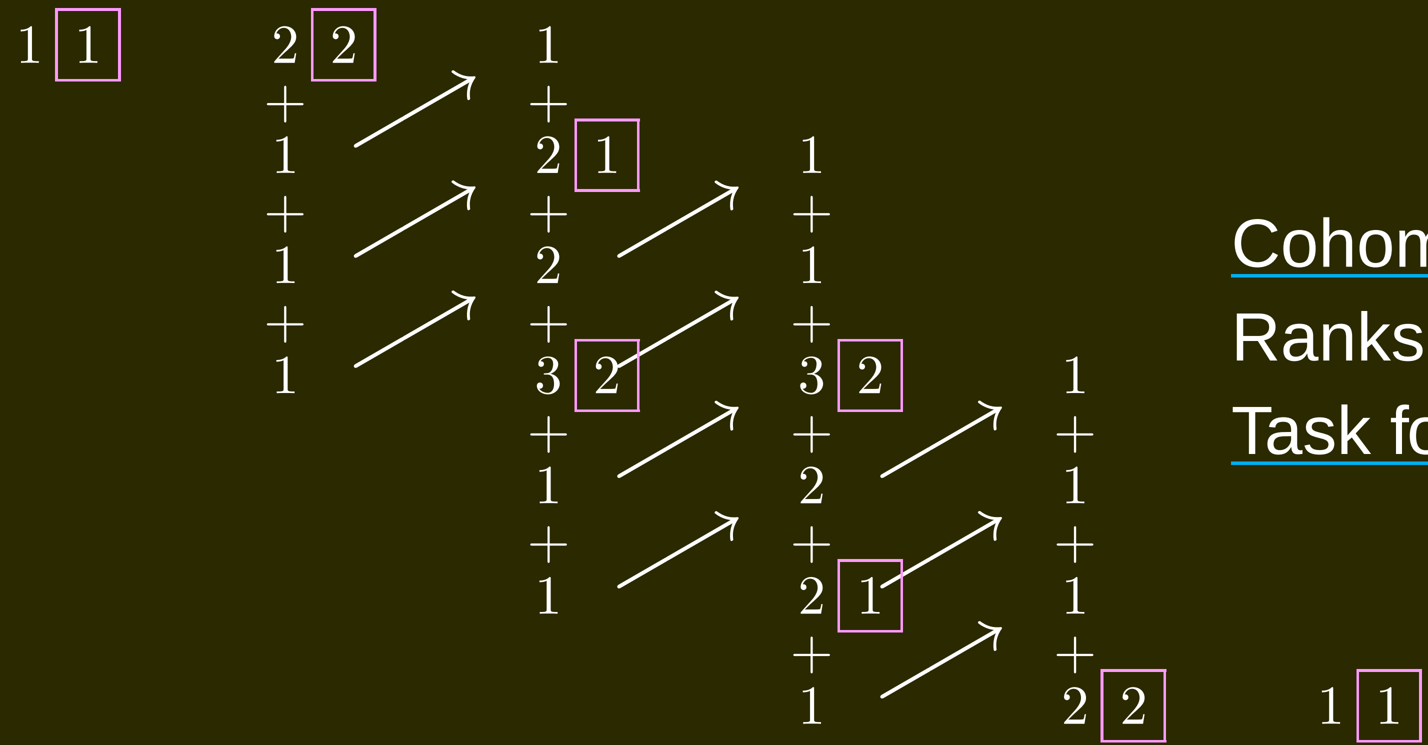
reduce using $\dfrac{\partial}{\partial s}$ and $\dfrac{\partial}{\partial t}$

Third Grushin complex

$$X = \frac{\partial}{\partial x} \qquad Y = x^3 \frac{\partial}{\partial y} \qquad Z \equiv [X, Y] \qquad W \equiv [X, Z] \qquad U \equiv [X, W]$$

Look for symmetry reduction from $\mathbb{R}^5$

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial r} + x\frac{\partial}{\partial s} + x^2\frac{\partial}{\partial t} + x^3\frac{\partial}{\partial y} \qquad \cdots$$



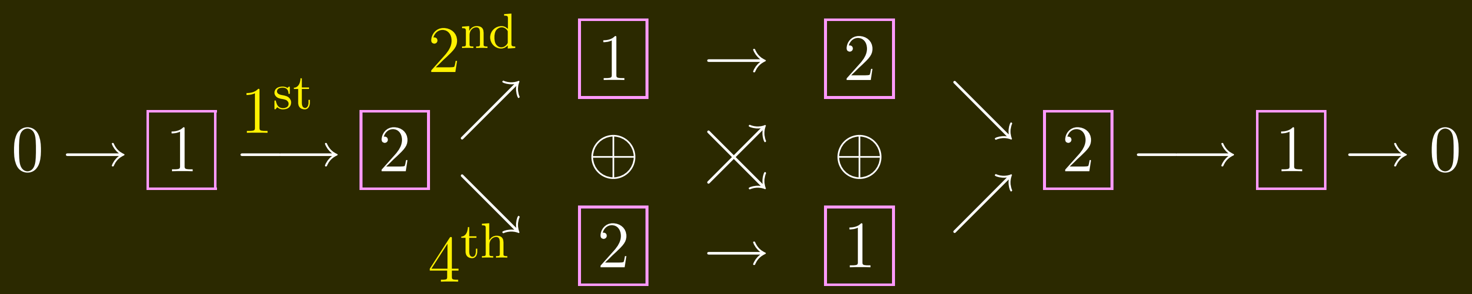
Cohomology?

Ranks=$1, 2, 3, 3, 2, 1$?

Task for babies?

Task for babies?

Yes! Everything works as expected $\rightsquigarrow$ a complex



reduce using $\frac{\partial}{\partial r}$ $\frac{\partial}{\partial s}$ $\frac{\partial}{\partial t}$

Integrability conditions

$$\left.\begin{array}{rcl} Xf &=& g \\ Yf &=& h \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} Y^2 g = YXh - Zh \\ X^4 h = X^3 Y g + X^2 Z g + XWg + Ug \\ XYZg - 2YWg + Z^2 g = X^2 Z h - 3XWh + 2Uh \end{array}\right.$$

Remark On $\mathbb{R}^2$ also $2YZg = 2XZh - 5Wh$

Conundrum: who ordered that?

ME, J Slovák, V Souček

Fourth Grushin

Not a task for babies!

Slow growth distribution $(2, 3, 4, 5, 6)$

Not a parabolic geometry: $\dfrac{\mathrm{SL}(2,\mathbb{R}) \ltimes \bigodot^4 \mathbb{R}^2}{\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}}$

Cf B Doubrov & A Medvedev

EG Lie algebra cohomology $\Rightarrow$ ranks are

$$1, 2, 3, 4, 3, 2, 1$$

and then

$$1, 2, 4, 6, 6, 4, 2, 1$$

for slow growth $(2, 3, 4, 5, 6, 7)$ et cetera

THE END

THANK YOU