Integral geometry on complex projective space

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Topics

- Funk or Radon transforms on S^2 or $\mathbb{R}^2$
- X-ray transform on $\mathbb{RP}_3$
- Penrose transform on $\mathbb{CP}_3$
- Other Penrose transforms
- Complex v. Real (complex methods)
- X-ray transform on $\mathbb{CP}_3$
- Real v. Complex (real methods)
- Other double fibration transforms

Funk-Radon

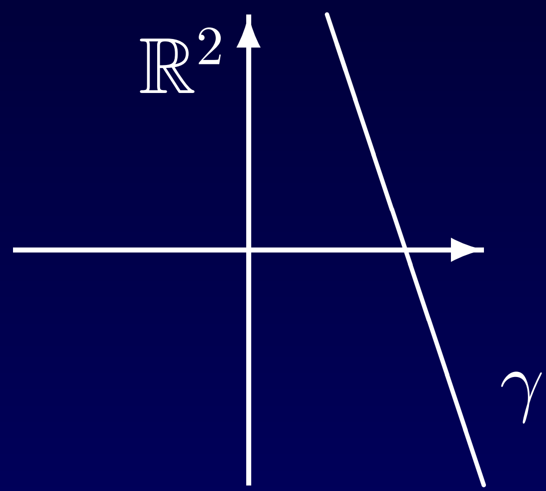
- Funk (1913)

$$f \in \Gamma(S^2, \mathcal{E})$$

$$\phi(\gamma) = \oint_\gamma f$$

- Radon (1917)

$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$



$$\phi(\gamma) = \int_\gamma f$$

Radon=Funk!

$$\begin{array}{ccc} \mathcal{F} : \Gamma_{\text{even}}(S^2, \mathcal{E}) & \xrightarrow{\simeq} & \Gamma_{\text{even}}(S^2, \mathcal{E}) \\ \| & & \| \\ \Gamma(\mathbb{RP}_2, \mathcal{E}) & & \Gamma(\mathbb{RP}_2, \mathcal{E}) \end{array}$$

Better: (bottom row of the diagram above)

Better still: $\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\simeq} \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1))$

Usual affine coördinates $\mathbb{R}^2 \hookrightarrow \mathbb{RP}_2 \quad \rightsquigarrow$

$$\left.\begin{array}{cc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} \\ \downarrow & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} \end{array}\right\} \text{ agree!}$$

John (1938)

The X-ray transform according to John

$$\Gamma_*(\mathbb{R}^3, \mathcal{E}) \ni f \mapsto \quad \phi(\gamma) = \int_\gamma f$$

Better: $\quad \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \ni f \mapsto \quad \phi(\gamma) = \oint_\gamma f$

NB invariance under $\mathrm{SL}(4, \mathbb{R})$ because

$$\Gamma(\mathbb{RP}_1, \mathcal{E}(-2)) \cong \Gamma(\mathbb{RP}_1, \Lambda^1) \xrightarrow{\int} \mathbb{R}$$

is invariant under $\mathrm{SL}(2, \mathbb{R})$.

X-ray transform on $\mathbb{RP}_3$

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1])$$

Range?

Theorem (John)

$$\phi = \mathcal{X} f \iff \Box\phi = 0$$

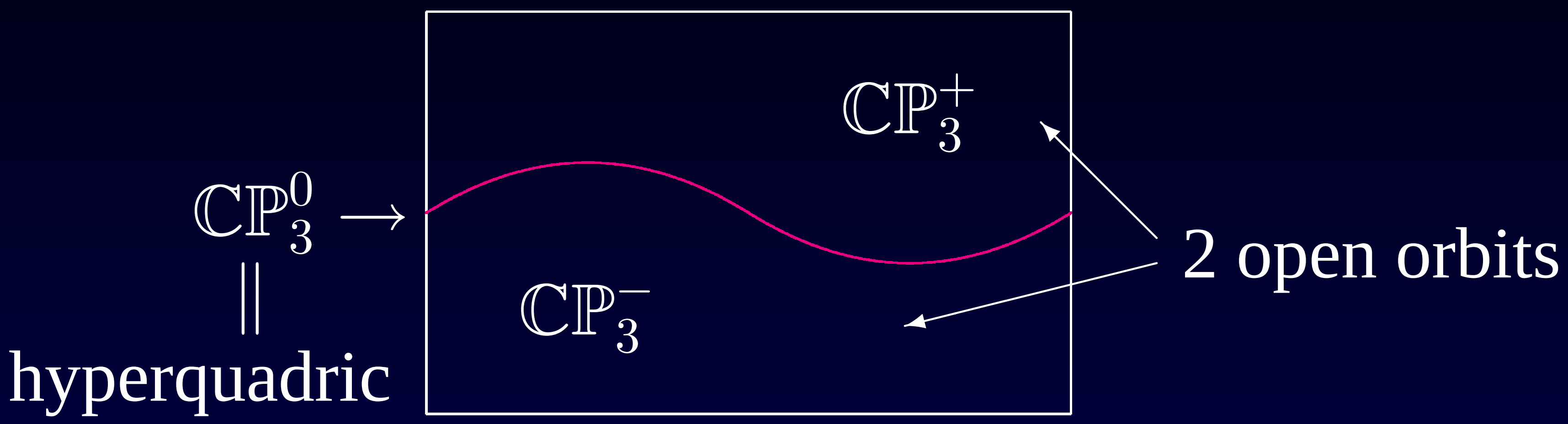
where

$$\Box : \widetilde{\mathcal{E}}[-1] \to \widetilde{\mathcal{E}}[-3]$$

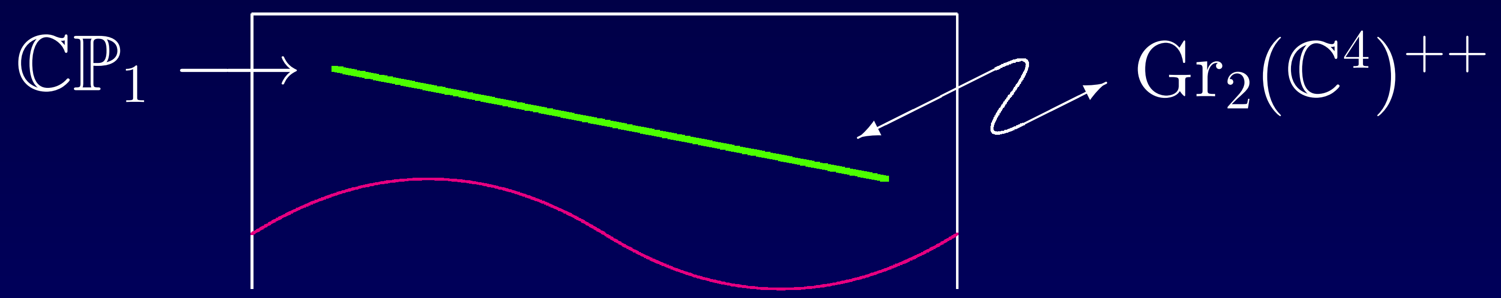
$=$ ultrahyperbolic wave operator ($\mathrm{SL}(4, \mathbb{R})$-invariant).

Penrose (1978–81)

$\mathrm{SU}(2,2)$ acting on $\mathbb{CP}_3$ has 3 orbits:–



$\mathrm{SU}(2,2)$ acting on $\mathrm{Gr}_2(\mathbb{C}^4)$ has 6 orbits (3 open):–



Penrose transform

$$\mathcal{P} : H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{C}^4)^{++}, \mathcal{O}[-1])$$

NB: $H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \cong H^1(\mathbb{CP}_1, \Omega^1) \xrightarrow{\;\int\;} \mathbb{C}$

Range?

Theorem (E-Penrose-Wells (1981))

$$\phi = \mathcal{P}\omega \iff \Box\phi = 0$$

where

$$\Box : \mathcal{O}[-1] \to \mathcal{O}[-3]$$

$=$ holomorphic wave operator ($\mathrm{SL}(4, \mathbb{C})$-invariant).

Twistor transform (cf. Fourier)

$$\mathcal{T} : H^1(\mathbb{CP}_3^+, \mathcal{O}(-2)) \xrightarrow{\cong} H^1(\mathbb{CP}_3^{*-}, \mathcal{O}(-2))$$

both $\cong$ {+ve-frequency hyperfunction solutions of the wave-equation on compactified Minkowski space}.

$$\begin{array}{ccc} \mathcal{T} : H^1(|\mathbb{CP}_3^+|, \mathcal{O}(-2)) & \xrightarrow{\cong} & H^1(|\mathbb{CP}_3^{*-}|, \mathcal{O}(-2)) \\ & & \downarrow\wr \\ & & H^2_*(\mathbb{CP}_3^{*+}, \mathcal{O}(-2)) \end{array}$$

$\cong$ real-analytic solutions. Defines unitary structure:–

$$\langle \alpha, \beta \rangle \equiv \int_{\mathbb{CP}_3^+} \alpha \wedge \overline{\mathcal{T}\beta}$$

Double fibration

$$\mathbb{F}_{1,2}(\mathbb{C}^4)$$

$$\mu \swarrow \qquad \searrow \nu$$

$$\mathbb{CP}_3 = \mathbb{F}_1(\mathbb{C}^4) \qquad\qquad \mathbb{F}_2(\mathbb{C}^4) = \mathrm{Gr}_2(\mathbb{C}^4)$$

fibres $\cong \mathbb{CP}_1$

$$\mathbb{CP}_3^+ = \mu(\nu^{-1}(\mathrm{Gr}_2(\mathbb{C}^4)^{++}))$$

Spectral sequence (cf. Leray)

$$E_1^{p,q} = \Gamma(\mathrm{Gr}_2(\mathbb{C}^4)^{++}, \nu_*^q(\Omega_\mu^p \otimes \mu^* V)) \Longrightarrow H^{p+q}(\mathbb{CP}_3^+, \mathcal{O}(V))$$

Twistor fibration (cf. Hopf)

$$\begin{array}{ll} \mathbb{CP}_3 & \mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \\ \tau \downarrow & \qquad\qquad \to \Gamma(U, \mathcal{E}[-1]) \\ S^4 \supseteq {}^{\text{open}} U & \end{array}$$

Range?

$$\phi = \mathcal{P}\omega \iff \Box\phi = 0$$

where

$$\Box : \mathcal{E}[-1] \to \mathcal{E}[-3]$$

$=$ conformal Laplacian ($\mathrm{SO}(5,1)$-invariant).

Other Penrose transforms

$$\begin{array}{c} Z \\ \tau\downarrow \\ M \end{array}$$

Atiyah-Hitchin-Singer (1978-81)

M $\leftarrow$ anti-self-dual 4-manifold

NB: a conformally invariant construction.

$$\mathcal{P} : H^1(Z, \kappa^{1/2}) \cong \{\phi \in \Gamma(M, \mathcal{E}[-1]) \text{ s.t. } \Box\phi = 0\},$$

where $\Box =$ conformal Laplacian $\Delta + R/6$.

$$\mathcal{P} : H^1(Z, \kappa) \cong \{\phi \in \Gamma(M, \Lambda^2_-) \text{ s.t. } d\phi = 0\}$$

$=$ anti-self-dual Maxwell fields on M.

Example (Fubini-Study)

$\mathbb{CP}_2$ (with opposite orientation) is anti-self-dual

$$\begin{array}{ccc} \mathbb{F}_{1,2}(\mathbb{C}^3) & \ni L \subset P \\ \tau\!\downarrow & \tau\!\downarrow \\ \mathbb{CP}_2 & \ni L^\perp \cap P \end{array}$$

Conclusions

$H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \mathcal{O}) = 0 \quad \Rightarrow \quad$ exactness of

$$\Gamma(\mathbb{CP}_2, \Lambda^0) \to \Gamma(\mathbb{CP}_2, \Lambda^1) \to \Gamma(\mathbb{CP}_2, \Lambda^2_+).$$

$H^1(\mathbb{F}_{1,2}(\mathbb{C}^3), \Theta) = 0 \quad \Rightarrow \quad$ exactness of

$$\Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_2, \textstyle\bigodot^2_\circ \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{CP}_2, \boxplus_{\circ +}).$$

Complex v. Real (complex methods)

$\dim_{\mathbb{R}} = 6$

$$\begin{array}{ccccccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & & & & \nu^{-1}(\mathrm{Gr}_2(\mathbb{R}^4)) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} & \supseteq & \swarrow & & \searrow \\ \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{C}^4) & & \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

$\rightsquigarrow$ fibration

$$\tau : \mathbb{CP}_3 \setminus \mathbb{RP}_3 \ni [Z] \longmapsto [iZ \wedge \bar{Z}] \in \mathrm{Gr}_2(\mathbb{R}^4)$$

Calculation $\Longrightarrow$

$$\Gamma(\mathrm{Gr}_2, \widetilde{\mathcal{E}}(-1)) \ni \phi \overset{\Box}{\longmapsto} 0 \iff \bar{\partial}\,\underbrace{\partial\tau^*\phi}_{\Gamma(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \Lambda^{0,1}(-2))} = 0 \text{ !!!}$$

But (see later)

- $H^0(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0,$
- $H^1(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0.$

Therefore, if $\square\phi = 0$ then

$$\boxed{\exists ! F \in \Gamma(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{E}(-2)) \ \text{ s.t. } \ \bar{\partial} F = \partial \tau^* \phi.}$$

Theorem (Bailey-E-Gover-Mason (1994–2003))

$$\phi = \mathcal{X} f \quad \Rightarrow \quad f = \text{`lim'} F.$$

$\lim \Re F = f \qquad \lim \Im F =$ Hilbert transform of f.

Real-analytic version (Aryapoor (2009)):–

$$\Gamma(\mathbb{RP}_3, \mathcal{O}(-2)) \xrightarrow{\simeq} H^1_*(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) \xrightarrow{\simeq}$$
$$\{\phi \in \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{O}}(-1)) \text{ s.t. } \square\phi = 0\}.$$

X-ray transform on $\mathbb{CP}_3$

$$
\begin{array}{lrl}
0) & \mathcal{X}:\Gamma(\mathbb{CP}_3,\Lambda^0)\ni f\mapsto & \gamma\mapsto\oint_\gamma f\\
1) & \mathcal{X}:\Gamma(\mathbb{CP}_3,\Lambda^1)\ni \omega\mapsto & \gamma\mapsto\oint_\gamma \omega\\
2) & \mathcal{X}:\Gamma(\mathbb{CP}_3,\bigodot^2\Lambda^1)\ni h_{ab}\mapsto & \gamma\mapsto\oint_\gamma h_{ab}\\
\vdots & \vdots & \vdots
\end{array}
$$

where $\gamma\in\{$Fubini-Study geodesics$\}$. Kernel?

- 0) $\mathcal{X}$ is injective
- 1) $\mathcal{X}\omega=0 \iff \omega=df$
- 2) $\mathcal{X}h_{ab}=0 \iff h_{ab}=\nabla_{(a}X_{b)}$
- $\vdots$ $\mathcal{X}h_{ab\cdots d}=0 \iff h_{ab\cdots d}=\nabla_{(a}X_{b\cdots d)}$

Tsukamoto (1981)

Real v. Complex (real methods)

- Bootstrap from $\mathbb{RP}_2$ to $\mathbb{CP}_2$ (E (1997))
- Bootstrap from $\mathbb{CP}_2$ to $\mathbb{CP}_3$ (Tsukamoto (1981))
- Bootstrap from $\mathbb{RP}_3$ to $\mathbb{CP}_3$ (E-Goldschmidt (…))

Geodesics? $\gamma \subset \mathbb{RP}_2 \subset \mathbb{CP}_2$ and act by $\mathrm{SU}(3)$.

Example

$$\begin{aligned} \oint_\gamma \omega = 0,\ \forall\,\gamma \quad &\Rightarrow \quad d\omega|_{\mathbb{RP}_2} = 0,\ \forall\,\mathbb{RP}_2 \\ &\Leftrightarrow \quad d\omega = f\,J \quad \text{(J=Kähler form)} \\ &\Rightarrow \quad d_+\omega = 0. \quad \text{However,} \end{aligned}$$

$\Gamma(\mathbb{CP}_2, \Lambda^0) \to \Gamma(\mathbb{CP}_2, \Lambda^1) \to \Gamma(\mathbb{CP}_2, \Lambda^2_+)$ is exact!

Proof of Tsukamoto

Similarly, to prove

$$\oint_\gamma h_{ab} = 0, \; \forall\, \gamma \Rightarrow h_{ab} = \nabla_{(a} X_{b)}$$

on $\mathbb{CP}_2$, we may use the real ingredients:–

- it's true on $\mathbb{RP}_2$,
- $\Gamma(\mathbb{CP}_2, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_2, \bigodot^2_\circ \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{CP}_2, \boxplus_{\circ +})$ is exact.

These, in turn, may be proved by complex methods!

More double fibrations (…, E-Wolf (…))

$G =$ complex simple Lie group e.g. $\mathrm{SL}(4,\mathbb{C})$

$G/P =$ complex flag manifold e.g. $\mathbb{CP}_3$

$G_0 =$ real form of G e.g. $\mathrm{SL}(4,\mathbb{R})$

$D =$ open orbit of G_0 on Z e.g. $\mathbb{CP}_3 \setminus \mathbb{RP}_3$

$\mathcal{M}_D =$ Wolf cycle space (Stein!) e.g. $\{$quadrics$\}$

$$\begin{array}{ccc} & \mathcal{X}_D & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ D & & \mathcal{M}_D \end{array} \quad \overset{\text{e.g.}}{\subset} \quad \begin{array}{ccc} & G/(K\cap P) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbb{CP}_3 & & G/K \end{array}$$

$K = \mathrm{SO}(4,\mathbb{C})$

$\overset{\text{e.g.}}{\Longrightarrow} H^r(\mathbb{CP}_3 \setminus \mathbb{RP}_3, \mathcal{O}(-2)) = 0$ for $r = 0, 1.$

As required, earlier.

THE END

THANK YOU!