Some Elliptic and Subelliptic Complexes from Geometry

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[based on joint work with Robert Bryant,
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De Rham complex

$$\boxed{\text{in } \mathbb{R}^3}$$

$$f \overset{\text{grad}}{\longmapsto} \nabla_i f \qquad \omega_i \overset{\text{curl}}{\longmapsto} \epsilon_i{}^{jk} \nabla_j \omega_k \qquad \phi_i \overset{\text{div}}{\longmapsto} \nabla^i \phi_i$$

$$\boxed{\text{on a smooth manifold}}$$

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \to 0$$

Elliptic and locally exact (mostly)

$$\Gamma(U, \Lambda^{p-1}) \xrightarrow{d} \Gamma(U, \Lambda^p) \xrightarrow{d} \Gamma(U, \Lambda^{p+1}) \quad \text{is exact} \quad p \geq 1$$

$$\ker : \Gamma(U, \Lambda^0) \xrightarrow{d} \Gamma(U, \Lambda^1) = \mathbb{R}$$

Coeffective complex

M = symplective manifold of dimension $2n$.
$J \in \Gamma(M, \Lambda^2)$ non-degenerate and $dJ = 0$.

$$0 \to \Lambda^k_\perp \to \Lambda^k \xrightarrow{J\wedge\_} \Lambda^{k+2} \to 0 \text{ defines } \Lambda^k_\perp \ \forall\, n \le k \le 2n.$$

NB

$$\begin{array}{ccccccccc} 0 & \to & \Lambda^k_\perp & \to & \Lambda^k & \xrightarrow{J\wedge\_} & \Lambda^{k+2} & \to & 0 \\ & & d_\perp \downarrow & & d \downarrow & & d \downarrow & & \\ 0 & \to & \Lambda^{k+1}_\perp & \to & \Lambda^{k+1} & \xrightarrow{J\wedge\_} & \Lambda^{k+3} & \to & 0 \end{array}$$

Hence (Bouche 1990)

$$\Lambda^n_\perp \xrightarrow{d_\perp} \Lambda^{n+1}_\perp \xrightarrow{d_\perp} \cdots \xrightarrow{d_\perp} \Lambda^{2n-2}_\perp \xrightarrow{d_\perp} \Lambda^{2n-1} \xrightarrow{d} \Lambda^{2n} \to 0$$

Elliptic and locally exact

Coeffective complex cont'd

NB $J \wedge \_ : \Lambda^k \to \Lambda^{k+2}$ is surjective for $n \leq k \leq 2n-2$
$J \wedge \_ : \Lambda^{n-1} \to \Lambda^{n+1}$ is an isomorphism
$J \wedge \_ : \Lambda^{k-2} \to \Lambda^k$ is injective for $2 \leq k \leq n$

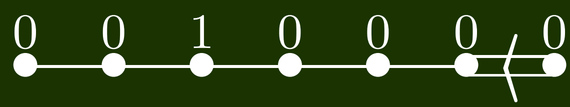
$0 \to \Lambda^{k-2} \xrightarrow{J\wedge\_} \Lambda^k \to \Lambda^k_\perp \to 0$ defines $\Lambda^k_\perp \; \forall\, 0 \leq k \leq n.$

Hence

$$0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \cdots \xrightarrow{d_\perp} \Lambda^{n-1}_\perp \xrightarrow{d_\perp} \Lambda^n_\perp$$

Elliptic

But $\Lambda^k_\perp \cong \Lambda^{2n-k}_\perp \; \forall\, 0 \leq k \leq n.$



Whence this is just the adjoint of the coeffective complex.

Coeffective complex cont'd cont'd

Thus, on a symplectic manifold

$$\begin{array}{ccccccccccccc} 0 & \to & \boxed{\Lambda^0} & \xrightarrow{d} & \boxed{\Lambda^1} & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d_\perp} & \cdots & \xrightarrow{d_\perp} & \Lambda^{n-1}_\perp & \xrightarrow{d_\perp} & \Lambda^n_\perp \\ & & & & & & & & & & & & \downarrow d^{(2)}_\perp \\ 0 & \leftarrow & \Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda^2_\perp & \xleftarrow{d_\perp} & \cdots & \xleftarrow{d_\perp} & \Lambda^{n-1}_\perp & \xleftarrow{d_\perp} & \Lambda^n_\perp \end{array}$$

Elliptic

$\boxed{}$ local cohomology $= \mathbb{R}$

Smith (1976, $n = 2$)
Rumin-Seshadri
Tseng-Yau

Global cohomology

$$\cdots \to H^r(M,\mathbb{R}) \to H^r_J(M) \to H^{r-1}(M,\mathbb{R}) \xrightarrow{J} H^{r+1}(M,\mathbb{R}) \to \cdots$$

Calibrated G_2-manifolds

M = smooth manifold of dimension 7
$J \in \Gamma(M, \Lambda^3)$ non-degenerate and $dJ = 0$.

$\Lambda^0 \xrightarrow{J\wedge\_} \Lambda^3$ and $\Lambda^1 \xrightarrow{J\wedge\_} \Lambda^4$ are injective
$\Lambda^2 \xrightarrow{J\wedge\_} \Lambda^5$ is an isomorphism
$\Lambda^3 \xrightarrow{J\wedge\_} \Lambda^6$ and $\Lambda^4 \xrightarrow{J\wedge\_} \Lambda^7$ are surjective

$$\begin{array}{ccccccccccc} 0 & \to & \boxed{\Lambda^0} & \to & \Lambda^1 & \to & \boxed{\Lambda^2} & \to & \Lambda^3_\perp & \to & \Lambda^4_\perp \\ &&&&&&&&&& \downarrow \\ 0 & \leftarrow & \Lambda^0 & \leftarrow & \Lambda^1 & \leftarrow & \Lambda^2 & \leftarrow & \Lambda^3_\perp & \leftarrow & \Lambda^4_\perp \end{array}$$

second order

Elliptic

$\boxed{}$ local cohomology $= \mathbb{R}$

Rumin complex

in $\mathbb{R}^3$ $\quad X, Y$ vector fields. Suppose $X, Y, Z \equiv [X, Y]$ span.

NB $\quad Xf = 0, Yf = 0 \Rightarrow f$ constant. $\quad$ Let $H \equiv \mathrm{span}\{X, Y\}$.

on a contact manifold $\quad H \subset TM \quad \leftrightsquigarrow \quad \Lambda^1 \twoheadrightarrow \Lambda^1_H$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & \Lambda^0 \arrow[r] \arrow[dr] & \Lambda^1 \arrow[d] \\ & & & \Lambda^1_H \end{tikzcd}$$

defines $\quad d_H : \Lambda^0 \to \Lambda^1_H$

s.t. $\mathbb{R} = \ker : \Lambda^0 \xrightarrow{d_H} \Lambda^1_H$

locally $\quad$ Darboux $\rightsquigarrow \quad [X, Z] = 0 = [Y, Z]$ wlg

$$\left.\begin{array}{rcl} Xf & = & g \\ Yf & = & h \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} XYg - X^2h + Zg & = & 0 \\ YXh - Y^2g - Zh & = & 0 \end{array}\right.$$

conversely? $\quad$ yes!

Rumin complex cont'd

$$
\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \xrightarrow{d} & \Lambda^4 & \xrightarrow{d} & \Lambda^5 \\
\| & & \| & & \| & & \| & & \| & & \| \\
\Lambda^0 & & \Lambda^1_H & & \Lambda^2_H & & \Lambda^3_H & & \Lambda^4_H & & \\
 & & + & \nearrow^{\text{injve}} & + & \nearrow^{\text{isosm}} & + & \nearrow^{\text{surve}} & + & & \\
 & & L & & \Lambda^1_H \otimes L & & \Lambda^2_H \otimes L & & \Lambda^3_H \otimes L & & \Lambda^4_H \otimes L
\end{array}
$$

Diagram chase (spectral sequence) $\rightsquigarrow$ $\boxed{\text{Subelliptic complex}}$

$$
\begin{array}{ccccccccccc}
\Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H & \xrightarrow{d_H} & \Lambda^2_{H\perp} & \xrightarrow{d_H^{(2)}} & \Lambda^2_{H\perp} \otimes L & \xrightarrow{d_H} & \Lambda^3_H \otimes L & \xrightarrow{d_H} & \Lambda^5 \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
0 \quad 0 \quad 0 & & -2 \quad 1 \quad 0 & & -3 \quad 0 \quad 1 & & -5 \quad 0 \quad 1 & & -6 \quad 1 \quad 0 & & -6 \quad 0 \quad 0 \\
\dim = 1 & & \dim = 4 & & \dim = 5 & & \dim = 5 & & \dim = 4 & & \dim = 1
\end{array}
$$

Parabolic geometry

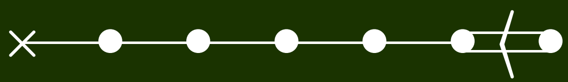
Geometries modelled on homogeneous spaces

$$\boxed{G/P} \qquad \begin{cases} G \text{ simple Lie group} \\ P \text{ parabolic subgroup} \end{cases}$$

Examples

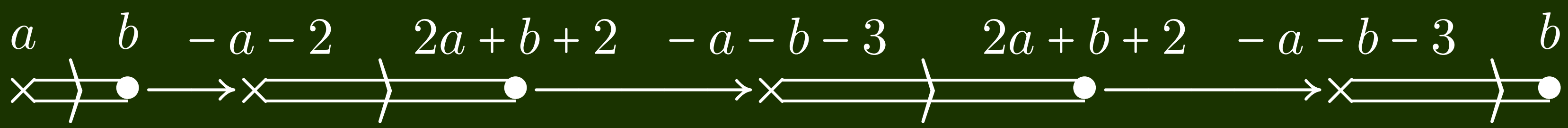
- conformal geometry $\mathrm{SO}(n+1,1)/P$
- CR geometry $\mathrm{SU}(n+1,1)/P$
- projective geometry $\mathrm{SL}(n+1,\mathbb{R})\Big/\left\{\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & * & \cdots & * \end{bmatrix}\right\}$
- contact projective geometry

$\mathrm{Sp}(2n+2,\mathbb{R})/P$

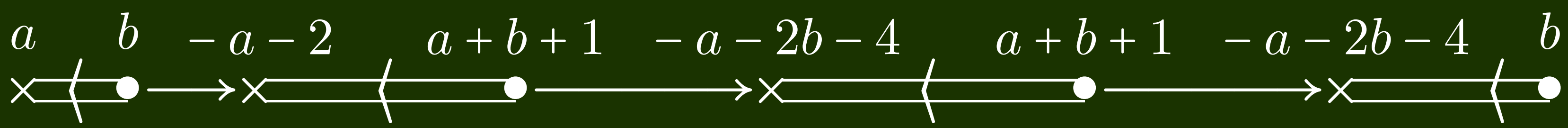


BGG complexes on the 3-sphere

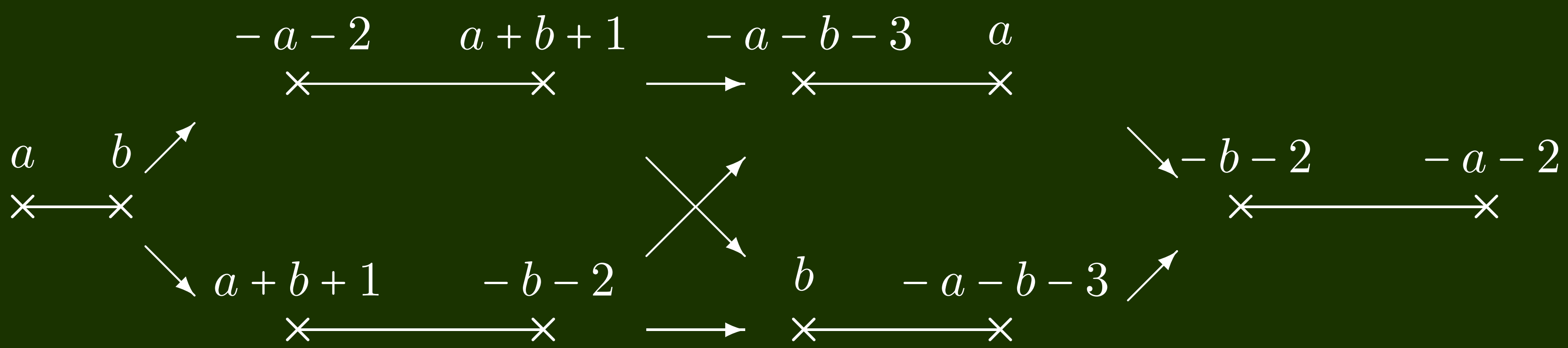
Conformal



Contact projective



CR



Applications: linear elasticity

displacement X_i $\boxed{\text{in } \mathbb{R}^3}$

$\downarrow$ Killing

strain $\nabla_i X_j + \nabla_j X_i$ Σ_{ij}

$\downarrow$ Saint-Venant

stress $\epsilon_i{}^{km}\epsilon_j{}^{\ell n}\nabla_k\nabla_\ell\Sigma_{mn}$ G_{ij}

$\downarrow$ Bianchi

load $\nabla^i G_{ij}$

BGG $\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet} \xrightarrow{\nabla^{(2)}} \overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet} \xrightarrow{\nabla} \overset{-5}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$ on 3-sphere

$\rightsquigarrow$ (Arnold-Falk-Winther) new stable finite element schemes

Applications: global analysis

$\boxed{\text{on a symplectic manifold}}$ Rumin-Seshadri complex

$$0 \to \boxed{\Lambda^0} \xrightarrow{d} \boxed{\Lambda^1} \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp \xrightarrow{d_\perp} \cdots$$

- $\Gamma(\mathbb{CP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp)$ exact
- $\Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \odot^2\Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{CP}_n, \boxplus_\perp \Lambda^1)$ exact

Killing — cf. St-Venant

$\boxed{\text{on a CR manifold}}$ Akahori-Garfield-Lee complex

$\boxed{\text{Hodge theory on symplectic manifolds}}$ Tseng-Yau

Further reading

- T. Akahori, P.M. Garfield, and J.M. Lee, *Deformation theory of 5-dimensional CR-structures and the Rumin complex*, Michigan Math. Jour. 50 (2002) 517–549.
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- R.L. Bryant, M.G. Eastwood, A.R. Gover, K. Neusser, *Some differential complexes within and beyond parabolic geometry*, arXiv:1112.2142.
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- M.G. Eastwood, *Extensions of the coeffective complex*, arXiv:1203.6714.
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- L.-S. Tseng and S.-T. Yau, *Cohomology and Hodge theory on symplectic manifolds: I and II*, arXiv:0909.5418 and arXiv:1011.1250.

THANK YOU

HAPPY BIRTHDAY NEIL!