Gaps Between Primes

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March 3, 2014
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   - Selberg Sieve

2. **Gaps Between Primes**
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   - Sketch Proof of Results

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   - General Prime Constellations
   - Distribution
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Theorem (Large Sieve Inequality)

For any complex sequence \((a_n)_{n \in \mathbb{N}}\); any \(M, N \in \mathbb{N}\) and all real \(Q > 1\)

\[
\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll (Q^2 + N) \sum_{n=M+1}^{M+N} |a_n|^2
\]
Definition (Coefficient Sequences)

A coefficient sequence is a finitely-supported sequence - whose support is a function of \(x\) - \(\alpha : \mathbb{N} \to \mathbb{R}\) such that

\[
|\alpha(n)| \ll \sigma(n)^{O(1)} (\log x)^{O(1)};
\]

If \(a \pmod{q}\) is a primitive residue class then define the signed discrepancy of \(\alpha\) by

\[
\Delta(\alpha; a, q) = \sum_{n \equiv a \pmod{q}} \alpha(n) - \frac{1}{\varphi(q)} \sum_{n,q=1} \alpha(n);
\]

If there exists some \(N \geq 1\) such that \(\alpha\) is supported on some interval of the form \([(1 - O((\log x)^{-A_0})N, (1 + O((\log x)^{-A_0}))N]\) then \(\alpha\) is said to be at scale \(N\);

If \(\alpha\) is at scale \(N\) then it is said to obey a Siegel-Walfisz theorem if

\[
\Delta(\alpha 1_{\cdot, q=1}; a, r) \ll \sigma(qr)^{O(1)} N (\log x)^{-A}.
\]
Theorem (Barban-Bombieri-Vinogradov)

Let $M$ and $N$ be such that $x \ll MN \ll x$ and $x^\epsilon \ll M, N \ll x^{1-\epsilon}$ for some fixed $\epsilon \in (0, 1)$. Let $\alpha$ and $\beta$ be coefficient sequences at scales $M$ and $N$ respectively. Suppose also that $\beta$ obeys a Siegel-Walfisz theorem. Then

$$\sum_{q \leq x^{1/2-o(1)}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta(\alpha \ast \beta; a, q)| \ll x(\log x)^{-A}$$

for some sufficiently slowly decaying $o(1)$ and any $A \in \mathbb{R}^+$.
Proposition (Heath-Brown Identity)

Let $K$ be some natural number. Then

$$\Lambda = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \mu^{K-j} \ast \mu_{\leq}^j \ast 1^{K-1} \ast \log$$

on $[1, 2x]$, where $\mu_{\leq}$ is the Möbius function restricted to $[1, (2x)^{1/K}]$. 
Conjecture (Elliott-Halberstam)

For all $\theta \in (0, 1)$

$$\sum_{q \leq x^{\theta}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \ll x(\log x)^{-A}.$$ 

If this holds for some $\theta \in (0, 1)$ then we say that the primes have level of distribution $\theta$. 
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Conjecture (Generalised Elliott-Halberstam)

For all $\theta \in (0, 1)$

$$\sum_{q \leq x^\theta} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta(\alpha \ast \beta; a, q)| \ll x(\log x)^{-A}$$
The Selberg lambda-square sieve is characterised by an expression of the form

$$\sum_{n} \left( \sum_{d|n,k} \Lambda_d \right)^2$$

where $\Lambda_n$ is some arithmetic function such that $\Lambda_1 = 1$ and $k$ is some number with a smoothness condition.
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Theorem (Goldston-Pintz-Yildirim)

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\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} = 0
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Theorem (Goldston-Pintz-Yildirim)

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Theorem (Maynard-Zhang)

\[ \liminf_{n \to \infty} (p_{n+1} - p_n) \ll 1 \]

Theorem (Maynard)

\[ \liminf_{n \to \infty} (p_{n+m} - p_n) \ll m^3 e^{4m} \]

The current best results are shown at the following website
with a timeline of results available
Theorem (Maynard)

Let $m \in \mathbb{N}$ and let $r \in \mathbb{N}$ be sufficiently large, depending on $m$. Also suppose that $\mathcal{A} = \{a_1, a_2, \ldots, a_r\}$ is a set of $r$ distinct integers. If $\mathcal{P}$ is the set of all $m$-tuples such that their translates are prime infinitely often then

$$\frac{\#\{\{h_1, \ldots, h_m\} \in \mathcal{A} : \{h_1, \ldots, h_m\} \in \mathcal{P}\}}{\#\{\{h_1, \ldots, h_m\} \in \mathcal{A}\}} \gg_m 1$$
Definition

Let us define the first and second GPY sums by

\[
S_1 = \sum_{\substack{N \leq n < 2N \\ n \equiv v_0 \pmod{W}}} \left( \sum_{d_i | n+h_i} \lambda_{d_1,\ldots,d_k} \right)^2
\]

\[
S_2 = \sum_{\substack{N \leq n < 2N \\ n \equiv v_0 \pmod{W}}} \left( \sum_{i=1}^{k} 1_{\mathbb{P}}(n+h_i) \right) \left( \sum_{d_i | n+h_i} \lambda_{d_1,\ldots,d_k} \right)^2
\]

where the \( h_i \) are the elements of some non-negative admissible set \( \mathcal{H} \), \( W = \log \log \log N \) and \( v_0 \) is the related to the open residue classes.
Let the primes have level of distribution $\theta \in (0, 1)$ and let $R = N^{\theta/2 - \delta}$ for some small, fixed $\delta \in \mathbb{R}^+$. Furthermore let $\lambda_{d_1, \ldots, d_k}$ be defined in terms of a fixed piecewise-differentiable function $F$ in the following way

$$
\lambda_{d_1, \ldots, d_k} = \left( \prod_{i=1}^{k} \mu(d_i)d_i \right) \sum_{\prod_{i=1}^{k} r_i = 1} \frac{\mu \left( \prod_{i=1}^{k} r_i \right)^2}{\prod_{i=1}^{k} \varphi(r_i)} F \left( \frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R} \right)
$$

whenever $\left( \prod_{i=1}^{k} d_i, W \right) = 1$ and $\prod_{i=1}^{k} d_i < r$ is square-free. Otherwise set $\lambda_{d_1, \ldots, d_k}$ equal to 0. Furthermore let $F$ be supported on $\mathcal{R}_k := \left\{(x_1, \ldots, d_k) \in [0, 1]^k : \sum_{i=1}^{k} x_i \leq 1 \right\}$. Then

$$
S_1 = \frac{(1 + o(1)) \varphi(W)^k N \log R^k}{W^{k+1}} I_k(F)
$$

$$
S_2 = \frac{(1 + o(1)) \varphi(W)^k N \log R^{k+1}}{W^{k+1} \log N} \sum_{m=1}^{k} J_k^{(m)}(F)
$$

provided that both $I_k(F)$ and $J_k^{(m)}(F)$ are non-zero for all $m$ where

$$
I_k(F) = \int_{0}^{1} \cdots \int_{0}^{1} F(t_1, \ldots, t_k)^2 dt_1 \cdots dt_k
$$

$$
J_k^{(m)}(F) = \int_{0}^{1} \cdots \int_{0}^{1} \left( \int_{0}^{1} F(t_1, \ldots, t_k) dt_m \right)^2 dt_1 \cdots dt_{m-1} dt_{m+1} \cdots dt_k
$$
Proposition

Suppose that $\mathcal{H}$ is an admissible set of size $k$ and let $S_k$ be the set of piecewise differentiable functions $F : [0, 1]^k \to \mathbb{R}$ supported on $\mathcal{R}_k$ with $I_k \neq 0$ and $J_k^{(m)} \neq 0$ for all $m$. Furthermore let

$$M_k = \sup_{F \in S_k} \left( \frac{\sum_{m=1}^{k} J_k^{(m)}(F)}{I_k(F)} \right), \quad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil.$$

Then there are infinitely many integers $n$ such that at least $r_k$ of the $n + h_i$ are prime. In particular

$$\liminf_{n \to \infty} \left( p_{n+r_k-1} - p_n \right) \leq \max_{1 \leq i, j \leq k} (h_i - h_j).$$
Theorem

Unconditionally

\[ \liminf_{n \to \infty} (p_{n+1} - p_n) \leq 600 \]

and, under the assumptions of the Elliott-Halberstam conjecture

\[ \liminf_{n \to \infty} (p_{n+1} - p_n) \leq 12 \]

\[ \liminf_{n \to \infty} (p_{n+2} - p_n) \leq 600 \]
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Conjecture (Polignac, 1843)

Let \( k \) be any positive integer. Then, for infinitely many \( n \in \mathbb{N} \), we have that \( p_{n+1} - p_n = 2k \).
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Let $k$ be any positive integer. Then, for infinitely many $n \in \mathbb{N}$, we have that $p_{n+1} - p_n = 2k$.

Conjecture (Dickson, 1904)

Let $a_1 + b_1 n, a_2 + b_2 n, \ldots, a_k + b_k n$ be a finite set of linear forms with integer coefficients where $b_i \geq 1$ for all $1 \leq i \leq n$. Then, if there is no positive integer $m$ divide all the products $\prod_{i=1}^{k} f_i(n)$ for all integers $n$ then there exist infinitely many natural numbers $n$ such that all of the linear forms are prime.
Conjecture (Hardy-Littlewood, 1923)

Let $P_2(n)$ be the number of prime pairs less than $n$. Then

$$P_2(n) \sim 2C_2 \frac{n}{(\log n)^2} \prod_{p \in \mathbb{P}\{2\}} \frac{p-1}{p-2}$$

where

$$C_2 = \prod_{\wp \in \mathbb{P}\{2\}} \left(1 - \frac{1}{(\wp - 1)^2}\right) \approx 0.66$$
Finally
Finally

\[
\liminf_{n \to \infty} (p_{n+1} - p_n) \geq 2
\]