Prime Numbers

How Far Apart Are They?

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1. Distribution of Prime Numbers
   - Behaviour of $\pi(x)$
   - Behaviour of $\pi(x; a, q)$
2. Prime Constellations
   - Distance Between Neighbouring Primes
   - Beyond Bounded Gaps
3. Diophantine Approximation
   - Classical Theory
   - Relation to Bounded Gaps

Basic Definitions

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- The number of coprime integers to n which do not exceed n is denoted by $\varphi(n)$.

Growth of $\pi(x)$

Theorem (Prime Number Theorem, Hadamard-de la Vallée Poussin, 1896)

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Putting $x + \log x$ into the top characterisation tells us that the asymptotic gap between primes is $\log x$.

Primes in Short Intervals

Theorem (Heath-Brown, 1988)

Let $\theta \in (7/12, 1)$. *Then*

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Theorem (Maynard, 2014)

Let $x, y > 1$, possibly dependent on each other. Then there are $\gg x \exp(-\sqrt{\log x})$ integers $x_0 \in [x, 2x]$ such that

$$\pi(x_0 + y) - \pi(x_0) \gg \log y. \tag{1}$$

Arithmetic Progressions

Theorem (Dirichlet, 1837)

Let $a, q \in \mathbb{N}$ be two coprime integers. Then the arithmetic progression

$$a, a + q, a + 2q, \ldots$$

contains infinitely many prime numbers.

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Theorem (Green-Tao, 2004)

The primes contain arbitrarily long arithmetic progressions.

Growth of $\pi(x; a, q)$

Theorem (Barban-Bombieri-Vinogradov, 1961-1987-1965)

For any small ϵ and real A

$$\sum_{q \leq x^{1/2-\epsilon}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \ll x (\log x)^{-A}.$$

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Conjecture (Elliott-Halberstam, 1968)

For all $\theta \in (0, 1)$

$$\sum_{q \leq x^{\theta}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \ll x(\log x)^{-A}.$$

If this holds for some $\theta \in (0, 1)$ then we say that the primes have level of distribution θ.

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Normalised Prime Gaps

Theorem (Westzynthius, 1931)

Let p_n be the nth prime number. Then

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log n} = \infty.$$

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Theorem (Golston-Pintz-Yildirim, 2006)

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Theorem (Banks-Freiman-Maynard, 2014)

The set of limit points of the sequence of normalised prime gaps contains 2% of all non-negative real numbers.

Bounded Prime Gaps

Conjecture (Twin Prime Conjecture)

There are infinitely many prime numbers p and q which differ by precisely 2. In other words

$$\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$$

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Theorem (Zhang, 2013)

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Theorem (Maynard-Zhang, 2013)

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 600$$

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Theorem (Maynard-Polymath-Zhang, 2013-2014)

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\leq 246$$

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Theorem (Maynard, 2014)

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \ll m^3 e^{4m}$$

Generalisations of the Bounded Gaps Conjecture

Conjecture (Polignac, 1843)

Let k be any positive integer. Then, for infinitely many $n \in \mathbb{N}$, we have that $p_{n+1} - p_n = 2k$. If this holds then $2k$ is called a Polignac number.

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Conjecture (Dickson, 1904)

Let $a_1 + b_1 n, a_2 + b_2 n, \ldots, a_k + b_k n$ be a finite set of linear forms with integer coefficients where $b_i \geq 1$ for all $1 \leq i \leq n$. Then, if there is no positive integer m divide all the products $\prod_{i=1}^{k} f_i(n)$ for all integers n then there exist infinitely many natural numbers n such that all of the linear forms are prime.

Partial Results Towards Polignac's Conjecture

Theorem (Pintz, 2013)

There is some ineffective constant c such that every interval of the form $[m, m+c]$ contains a Polignac number.

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The set of Polignac numbers contains arbitrarily long arithmetic progressions.

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Corollary (Hanson, 2014)

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Theorem (Hanson, 2014)

Every arithmetic progression of the form $q, 2q, 3q, \ldots$ contains infinitely many Polignac numbers.

Approximating Reals with Rationals

Theorem (Dirichlet, 1834)

Let x be a real number. Then there are infinitely many coprime m, n such that

$$\left| x - \frac{m}{n} \right| < \frac{1}{n^2}$$

Approximating Reals with Rationals

Theorem (Hurwitz, 1891)

Let x be a real number. Then there are infinitely many coprime m, n such that

$$\left| x - \frac{m}{n} \right| < \frac{1}{\sqrt{5} n^2}$$

Bad

Definition (Badly Approximable Number)

A number x is called badly approximable if there is some $c \in \mathbb{R}^+$ such that

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for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set of all badly approximable is written Bad.

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Theorem (Jarnik, 1928)

Bad *has Hausdorff dimension* 1 *and measure* 0*.*

Prime Approximation

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Fix $N \in \mathbb{N}$. Then there are infinitely many prime numbers p, q such that

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Conjecture (Hanson-Haynes)

Fix some $\alpha \in \mathbb{R}^+$ and $N \in \mathbb{R}_{\geq 0}$. Then there are infinitely many prime numbers p, q such that

$$|p - \alpha q| \leq N$$