Prime Numbers
How Far Apart Are They?

Stijn S.C. Hanson

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1 Distribution of Prime Numbers
   • Behaviour of $\pi(x)$
   • Behaviour of $\pi(x; a, q)$

2 Prime Constellations
   • Distance Between Neighbouring Primes
   • Beyond Bounded Gaps

3 Diophantine Approximation
   • Classical Theory
   • Relation to Bounded Gaps
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Basic Definitions

- A prime number is any positive integer bigger than 1 whose only divisors are 1 and itself;
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- Two integers $n$ and $m$ are said to be coprime if their greatest common divisor is 1. I.e. if $(n, m) = 1;$
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- Two integers $n$ and $m$ are said to be coprime if their greatest common divisor is 1. I.e. if $(n, m) = 1$;
- The number of coprime integers to $n$ which do not exceed $n$ is denoted by $\varphi(n)$. 
Growth of $\pi(x)$

**Theorem (Prime Number Theorem, Hadamard-de la Vallée Poussin, 1896)**

\[
\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)
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Putting $x + \log x$ into the top characterisation tells us that the asymptotic gap between primes is $\log x$. 
Primes in Short Intervals

Theorem (Heath-Brown, 1988)

Let \( \theta \in (7/12, 1) \). Then

\[
\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x}
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Primes in Short Intervals

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Theorem (Maynard, 2014)

Let \( x, y > 1 \), possibly dependent on each other. Then there are \( \gg x \exp(-\sqrt{\log x}) \) integers \( x_0 \in [x, 2x] \) such that

\[
\pi(x_0 + y) - \pi(x_0) \gg \log y.
\]
Arithmetic Progressions

Theorem (Dirichlet, 1837)

Let $a, q \in \mathbb{N}$ be two coprime integers. Then the arithmetic progression

$$a, a + q, a + 2q, \ldots$$

contains infinitely many prime numbers.

Theorem (Green-Tao, 2004)
The primes contain arbitrarily long arithmetic progressions.
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Growth of $\pi(x; a, q)$

Theorem (Barban-Bombieri-Vinogradov, 1961-1987-1965)

For any small $\epsilon$ and real $A$

$$\sum_{q \leq x^{1/2-\epsilon}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \ll x(\log x)^{-A}.$$
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**Conjecture (Elliott-Halberstam, 1968)**

For all $\theta \in (0, 1)$

$$\sum_{q \leq x^{\theta}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \ll x(\log x)^{-A}.$$

If this holds for some $\theta \in (0, 1)$ then we say that the primes have level of distribution $\theta$. 

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Normalised Prime Gaps

**Theorem (Westzynthius, 1931)**

Let $p_n$ be the $n$th prime number. Then

$$\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} = \infty.$$
Normalised Prime Gaps

**Theorem (Westzynthius, 1931)**

Let $p_n$ be the $n$th prime number. Then

$$\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} = \infty.$$ 

**Theorem (Golston-Pintz-Yildirim, 2006)**

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} = 0$$
Normalised Prime Gaps

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Theorem (Banks-Freiman-Maynard, 2014)

*The set of limit points of the sequence of normalised prime gaps contains 2% of all non-negative real numbers.*
Conjecture (Twin Prime Conjecture)

There are infinitely many prime numbers $p$ and $q$ which differ by precisely 2. In other words

$$\liminf_{n \to \infty} (p_{n+1} - p_n) = 2$$
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**Theorem (Zhang, 2013)**

\[
\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 70,000,000
\]
Bounded Prime Gaps

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$$\liminf_{n \to \infty} (p_{n+1} - p_n) = 2$$

Theorem (Maynard-Zhang, 2013)

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 600$$
Bounded Prime Gaps

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Theorem (Maynard-Polymath-Zhang, 2013-2014)

\[
\lim \inf_{n \to \infty} (p_{n+1} - p_n) \leq 246
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Theorem (Maynard, 2014)

$$\liminf_{n \to \infty} (p_{n+m} - p_n) \ll m^3 e^{4m}$$
Generalisations of the Bounded Gaps Conjecture

**Conjecture (Polignac, 1843)**

Let $k$ be any positive integer. Then, for infinitely many $n \in \mathbb{N}$, we have that $p_{n+1} - p_n = 2k$. If this holds then $2k$ is called a Polignac number.
Conjecture (Polignac, 1843)

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Conjecture (Dickson, 1904)

Let $a_1 + b_1 n, a_2 + b_2 n, \ldots, a_k + b_k n$ be a finite set of linear forms with integer coefficients where $b_i \geq 1$ for all $1 \leq i \leq n$. Then, if there is no positive integer $m$ divide all the products $\prod_{i=1}^{k} f_i(n)$ for all integers $n$ then there exist infinitely many natural numbers $n$ such that all of the linear forms are prime.
Partial Results Towards Polignac’s Conjecture

Theorem (Pintz, 2013)

There is some ineffective constant \( c \) such that every interval of the form \([m, m + c]\) contains a Polignac number.
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Corollary (Hanson, 2014)

The set of Polignac numbers contains arbitrarily long arithmetic progressions.
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**Corollary (Hanson, 2014)**

The set of Polignac numbers contains arbitrarily long arithmetic progressions.

**Theorem (Hanson, 2014)**

Every arithmetic progression of the form \( q, 2q, 3q, \ldots \) contains infinitely many Polignac numbers.
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Approximating Reals with Rationals

Theorem (Dirichlet, 1834)

Let $x$ be a real number. Then there are infinitely many coprime $m, n$ such that

$$\left| x - \frac{m}{n} \right| < \frac{1}{n^2}$$
Theorem (Hurwitz, 1891)

Let $x$ be a real number. Then there are infinitely many coprime $m, n$ such that

$$\left| x - \frac{m}{n} \right| < \frac{1}{\sqrt{5}n^2}$$
Definition (Badly Approximable Number)

A number $x$ is called badly approximable if there is some $c \in \mathbb{R}^+$ such that

$$\left| x - \frac{m}{n} \right| \geq \frac{c}{n^2}$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set of all badly approximable is written $\text{Bad}$.
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Theorem (Jarnik, 1928)

\( \text{Bad} \) has Hausdorff dimension 1 and measure 0.
Prime Approximation

Theorem (Bounded Gaps Conjecture)

Fix $N \in \mathbb{N}$. Then there are infinitely many prime numbers $p, q$ such that

$$|p - q| \leq N$$
Prime Approximation

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Fix $N \in \mathbb{N}$. Then there are infinitely many prime numbers $p, q$ such that

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**Conjecture (Hanson-Haynes)**

Fix some $\alpha \in \mathbb{R}^+$ and $N \in \mathbb{R}_{\geq 0}$. Then there are infinitely many prime numbers $p, q$ such that

$$|p - \alpha q| \leq N$$