Definition

Let $G$ be a group and $V$ be a complex vector space. Then $\rho : G \to \text{GL}(V)$ is a representation if it is a homomorphism. Denote a representation by the pair $(\rho, V)$. A representation is irreducible if there are no non-trivial $\rho$-invariant subspaces of $V$. I.E. for all $U$ such that $0 \neq U \subsetneq V$. Write the set of all equivalence classes (under some relation) of irreducible representations of $G$ as $\hat{G}$. 

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Stijn Hanson (York) Group Representation Theory on Mixing Times
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Let $p$ be some probability measure on a group $G$. We define the Fourier transform of $p$ at the representation $(\rho, V)$ by

$$\hat{p}(\rho) = \sum_{g \in G} p(g)\rho(g)$$
**Theorem (Fourier Inversion Theorem)**

Let $p$ be a probability measure on a group $G$ and $\hat{p}$ be its Fourier transform. Then

$$p(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \text{tr}(\hat{p}(\rho) \rho(g^{-1}))$$
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Theorem (Plancherel’s Theorem)

Let $p$ and $q$ be some probability measures on a group $G$. Then

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\sum_{g \in G} p(g^{-1})q(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_{\rho} \text{tr}(\hat{p}(\rho) \hat{q}(\rho))
$$
We can show that the matrix associated to any representation \((\rho, V)\) is unitary. Thus

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where \(*\) represents the adjoint. By taking \(q(g) = p(g^{-1})^*\) we get

**Theorem (Plancherel (kinda))**

\[
\sum_{g \in G} |p(g)|^2 = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \text{tr}(\hat{p}(\rho)\hat{p}(\rho^*))
\]
Theorem

Let \( p \) be some probability measure on a finite group \( G \) and \( u \) be the uniform measure. Then

\[
|G| \sum_{g \in G} |p_t(g) - u(g)|^2 = \sum_{\rho \in \hat{G}}^* d_\rho \text{tr}(\hat{p}(\rho)^t(\hat{p}(\rho)^t)^*)
\]

where the sum is over all non-trivial irreducible representations.
We can use this to bound total variation distance with the simple observations that

\[ \| p(t)(x) - u \|_{TV}^2 \leq \frac{|G|}{4} \sum_{g \in G} |p(t)(g) - u(g)|^2 \]

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by the Cauchy-Schwarz inequality and Pythagoras’ theorem respectively. These give

\[ \frac{1}{4|G|} \sum_{\rho \in \hat{G}}^* d_\rho \text{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*) \leq \| p^{(t)} - u \|_{TV}^2 \leq \frac{1}{4} \sum_{\rho \in \hat{G}}^* d_\rho \text{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*) \]

or, slightly more vaguely,

\[ \| p^{(t)} - u \|_{TV}^2 \asymp \sum_{\rho \in \hat{G}}^* d_\rho \text{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*) \]
Suppose that $G$ is a group, $V$ is a finite-dimensional vector space and $(\rho, V)$ is a representation of $G$. For each $g \in G$ consider the matrix representation of $\rho(g)$, denoted $\rho(g)$, relative to some fixed basis in $V$. Define a function $\chi : G \rightarrow \mathbb{C}$ by $\chi(g) = \text{tr} \, \rho(g)$ for all $g \in G$. $\chi$ is called the character of $(\rho, V)$.
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It turns out that, if $G$ is Abelian, the characters form a group. Call this the dual group of $G$ and write it as $\tilde{G}$. Moreover $\tilde{G} \cong G$. 
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Definition

Let $G$ be a group and $\chi$ be the character of some irreducible representation of $G$. Then define the Fourier transform of some measure $p$ at $\chi$ as

$$\hat{p}(\chi) = \sum_{g \in G} p(g) \chi(g)$$
From now on we will take $G$ to be Abelian. Then $G \cong \tilde{G}$ implies that the collection $(\hat{p}(\chi))_{\chi \in \tilde{G}}$ is precisely the spectrum of $p$ viewed as a convolution operator.
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**Theorem**

\[ \rho \text{ is an irreducible representation if and only if } d_\rho = 1. \]

we get

\[ |G| \sum_{g \in G} |p^{(t)}(g) - u(g)|^2 = \sum_{\chi \in \tilde{G}}^* |\hat{p}(\chi)|^{2t} \]
Consider $G = \mathbb{Z}_n$. We know that $G$ is Abelian and so there are $n$ irreducible representations given by

$$\rho_j(N) = \left( e^{2\pi ij N/n} \right)$$

for any $N \in \mathbb{Z}_n$, $\times \in \mathbb{C}^*$ and $0 \leq j < n$. 
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Consider the simple random walk where $p(+1) = p(-1) = 1/2$. Then

$$\hat{p}(\chi_j) = \frac{1}{2}(\chi(+1) + \chi(-1))$$

$$= \frac{1}{2} \left( e^{t\pi ij/n} + e^{-t\pi ij/n} \right)$$

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Thus

$$\|p(t) - u\|_{TV}^2 \approx \sum_{j=1}^{n-1} |\cos(2\pi ij/n)|^{2t}$$