Analytic surgery and gluing formulae for determinants

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Analytic surgery (Adiabatic limit)

Let M be a closed manifold with an embedded hypersurface H. In topology, a basic technique is to cut M at H, resulting in a manifold with boundary $\overline{M}$ (possibly consisting of two components). One can then glue in a new manifold with boundary H, thus obtaining a new manifold.

In the study of global invariants such as the index, determinant, spectral flow, etc, a similar procedure is very useful. Here M is typically a Riemannian manifold. For the purposes of analysis it is useful to think of stretching M cylindrically across H, keeping the cross section fixed. In the limit one obtains a manifold with cylindrical ends (again, possibly with two components). This procedure is called *analytic surgery* or taking an *adiabatic limit.* (One could also just cut the manifold at H, but this precludes the possibility, irresistible to analysts, of taking a *limit* as the length of the neck goes to infinity.)

One way to do this is to insert a cylinder $H \times [-R, R]$ of length $2R$ near H, and let $R \to \infty$. The value of some global invariant, such as a determinant, on M_R can be compared to the same invariant on the two manifolds $M_{1,R}$ and $M_{2,R}$ which are the manifolds with boundary M_1, M_2 with a cylinder of length R attached. Suitable boundary conditions need to be imposed at the boundaries of $M_{1,R}$ and $M_{2,R}$. This is the approach of Douglas-Wojciechowski which they called the 'adiabatic limit'.

Another way, developed by McDonald and Mazzeo-Melrose, and developed further by myself in my thesis, is to write down a one-parameter family of metrics on M whose limit is a b-metric on $M \setminus H$. Let (x, y) be local coordinates near H such that H is given by $\{x = 0\}$. An example is

$$g_\epsilon = \frac{dx^2}{x^2 + \epsilon^2} + h(x, y, dy),$$

where $h(0, y, dy)$ is a metric on H. As $\epsilon \to 0$, this approaches the b-metric $dx^2/x^2 + h$ on $M \setminus H$. (Away from H we assume that g_ϵ remains nondegenerate as $\epsilon \to 0$.) The length of the neck is asymptotically $2\sinh^{-1}(1/\epsilon)$, so to compare with the previous scenario we should take $R = \sinh^{-1}(1/\epsilon)$.

One advantage of this procedure is that it eliminates the need to consider boundary conditions. On the other hand, the limiting manifold $\overline{M}$ has a b-metric, therefore has continuous spectrum which means that even defining global invariants such as the determinant is a challenge.

Determinants

Let M be a closed Riemannian manifold and let Δ be a (positive) Laplace-type operator on M, acting on sections of a vector bundle V. By Laplace-type I mean that Δ is a second order differential operator and the principal symbol takes the form

$$\sigma(\Delta)(x,\xi) = |\xi|^2_{g_x} \cdot \mathrm{Id},$$

where Id is the identity operator on V. We'll also assume that $\Delta = A^*A$ for some first order operator A, hence Δ is nonnegative and self-adjoint. Typical examples are the scalar Laplacian, $\Delta = (d+\delta)^2$ acting on forms on M, or $\Delta = \eth_-\eth_+$ acting on sections of the positive spin bundle over a spin manifold M.

The (modified) determinant $\det'(\Delta)$ is a regularization of the product of the eigenvalues of Δ, which is always divergent since the eigenvalues themselves tend to infinity. We first define the zeta function:

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s},$$

where λ_j are eigenvalues of Δ counted w. multiplicity.

Formally,

$$-\zeta'(0) = \sum_{\lambda_j} \lambda_j^{-s} \log \lambda_j \Big|_{s=0} = \sum \log \lambda_j = \log \prod \lambda_j$$

is the logarithm of the product of the nonzero eigenvalues of Δ. The remarkable fact is that $\zeta(s)$ continues meromorphically to $s = 0$ and is analytic there, so that we can *define* the modified determinant to be $-\zeta'(0)$. To see this you write the zeta function in terms of the heat kernel:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{tr}\left(e^{-t\Delta} - \Pi_0\right) \frac{dt}{t}$$

which follows directly from

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\lambda} \frac{dt}{t}.$$

The integral from 0 to ∞ is in danger of diverging at both endpoints. However, $\operatorname{tr} e^{-t\Delta} - \Pi_0$ decays exponentially at $t = \infty$ so the integral always converges at infinity. As $t \to 0$, it is well known that the heat kernel has an expansion

$$\operatorname{tr} e^{-t\Delta} \sim \sum_{j=0}^{k} t^{-n/2+j} a_j + e_k(t),\ e_k(t) = o(t^{-n/2+k}),$$

and if we substitute this expansion in at $t = 0$ we see that at worst the zeta function has simple poles at $s = n/2, n/2-1, n/2-2, \ldots$. Finally, $\Gamma(s)$ has a pole at $s = 0$ so the $1/\Gamma(s)$ factor ensures that the zeta function is regular at $s = 0$.

Determinants and the b-calculus

Now consider a manifold with boundary, $\overline{M}$, with a b-metric. Then any Laplace-type operator has continuous spectrum, so the zeta function cannot be defined via the eigenvalues. Moreover, the heat kernel is not trace class, so it cannot be defined using the trace of the heat kernel either. However, we can define the 'b-determinant' obtained by replacing the trace by the b-trace. That is, the b-determinant of Δ is given by $-{}^b\zeta'(0)$, where ${}^b\zeta(s)$ is the analytic continuation of

$$\frac{1}{\Gamma(s)}\int_0^1 t^s \,\text{b-tr}\, e^{-t\Delta}\,\frac{dt}{t} + \frac{1}{\Gamma(s)}\int_1^\infty t^s \,\text{b-tr}\, e^{-t\Delta}\,\frac{dt}{t}.$$

Again ${}^b\zeta$ continues meromorphically and is analytic at $s = 0$. What, if anything, is the b-determinant good for?

To answer this, consider the behaviour of the determinant under analytic surgery.

Analytic surgery — easy case

Let us try to find a manifold with corners on which the heat kernel $e^{-t\Delta_\epsilon}$ has simple behaviour.

For the 'single space', we may take $M \times [0, \epsilon_0]$ and blow up the set $H \times \{0\}$ where the metric is singular. This leads to a space where at $\epsilon = 0$ we have two boundary hypersurfaces, namely $\overline{M}$ and the cylinder $H \times \mathbb{R}$. We try to construct kernels, such as the resolvent or heat kernel of Δ, on the corresponding 'double space'. In order that the determinant has reasonable behaviour it is necessary to make the following assumption:

A1. The dimension of the null space of Δ_ϵ is constant as $\epsilon \to 0$.

We also temporarily make the following assumption:

A2. The induced Laplacian on the cross section, Δ_H, is invertible.

This is the easy case referred to in the heading. Under this assumption, the heat kernel can be constructed on the double space. This leads to the following result:

Theorem 1. *Under assumptions A1 and A2, the determinant has the following asymptotic expansion as* $\epsilon \to 0$ *(recall that* $R = \sinh^{-1}(1/\epsilon)$ *is the asymptotic length of the neck):*

$$\log\det\Delta_\epsilon = -2R\zeta_{\Delta_H}(-\frac{1}{2}) + \log \text{b-det}\,\Delta_{\overline{M}} + o(1).$$

[Here, we have assumed $\dim M$ *is odd.]*

This result is essentially due to Mazzeo-Melrose.

Analytic surgery — small eigenvalues

A more challenging problem is to suppress assumption A2 and allow null space on the cross section. The continuous spectrum on $\overline{M}$ starts from the smallest eigenvalue of Δ_H so assumption A2 means that the continuous spectrum starts at some positive number, hence the spectrum is discrete near zero. This makes the long time behaviour of the heat kernel rather simple as $t \to \infty$. However, if A2 fails, this is no longer true: continuous spectrum comes down to zero, hence there are small eigenvalues $\lambda_j(\epsilon)$ accumulating at zero and giving rise to interesting long time behaviour of the heat kernel.

It turns out that the single space above is not adequate to deal with this case. The main reason is that the rescaled distance function along the neck is not smooth on this space. By rescaled distance function, I mean (dist to H)/R. This is an important function because the small eigenvalues can be written in terms of it; roughly speaking, the eigenfunctions corresponding to small eigenvalues looks like

$$\cos(\pi j d/R) \cdot \phi(y), \quad \Delta_H \phi = 0.$$

We can get such a space by performing two more blowups, the first being a 'logarithmic' blowup and the second blowing up the corner. This is the 'logarithmic single space'. Then the resolvent and heat kernels can be constructed on the corresponding logarithmic double space. This sequence of blowups resolves the singularities of these kernels as $\epsilon \to 0$, and (I am pretty sure) no simpler space does the job.

It turns out that there is a one-dimensional spectral problem that controls the asymptotics of small eigenvalues. Namely, letting $\overline{M} = M_- \cup M_+$, there are subspaces $\Lambda^D_\pm$, of $\operatorname{null} \Delta_H$ which is the set of limiting values of smooth (but not L^2) solutions u of $\Delta_{M_\pm} u = 0$. Let $\Lambda^N_\pm$ be the orthogonal complement of $\Lambda^D_\pm$. Consider the operator

$$RN(\Delta)v = -\frac{d^2}{ds^2}v, \quad v(s) \in \operatorname{null} \Delta_H, \quad s \in [-1, 1],$$

with boundary values

$$v(\pm 1) \in \Lambda^D_\pm, \quad \partial_s v(\pm 1) \in \Lambda^N_\pm.$$

Let $\{z_j^2\}$ be the eigenvalues of this operator, which I denote RN(Δ). Then the small eigenvalues $\lambda_j(\epsilon)$ are either zero or $R^{-2}z_j^2 + O(R^{-3})$. This was proved by Müller and by myself, Mazzeo and Melrose independently, and perhaps by others as well.

Theorem 2. *Under the assumption A1, the determinant has the behaviour under analytic surgery*

$$\log\det\Delta_\epsilon = -2R\zeta_{\Delta_H}(-\frac{1}{2}) + \log R.\dim\operatorname{null}\operatorname{RN}(\Delta)$$
$$+\log \text{b-det}\,\Delta_{\overline{M}} + \log\det\operatorname{RN}(\Delta) + o(1).$$

This works particularly nicely with analytic torsion. In my thesis, I was able to prove (from first principles)

Theorem 3. *The b-analytic torsion ${}^bT(\overline{M}, E)$ of a flat bundle E over a manifold with boundary $\overline{M}$ satisfies*

$${}^bT(\overline{M}, E) = \tau(\overline{M}, E)2^{-1/4\chi_E(\partial M)}.$$

Determinant on manifold with long end

It is of interest to compare the determinant of a manifold with boundary with a long end $H \times [0, R]$ attached, with the b-determinant of the same manifold.

Theorem 4. (to be checked!) *Let M be a manifold with boundary H with product structure near the boundary, and let $M_R = M \cup H \times [0, R]$. Let Δ_R denote a Laplace-type operator on M_R subject to APS boundary conditions and let Δ_∞ denote the same operator on the manifold with infinite cylindrical end attached. Then as $R \to \infty$*

$$\log \det \Delta_R = -R\zeta_{\Delta_H}(-\frac{1}{2}) + \log \text{b-det}\, \Delta_\infty$$
$$-\frac{\log 2}{2}\zeta_{\Delta_H}(0) + o(1).$$

This follows by doubling the manifold and combining the result above with results of Park-Wojciechowski.