The b-calculus and index theory on manifolds with boundary

Andrew Hassell

Manifolds with boundary

Let X be a compact n-dimensional manifold with boundary, $\partial X = Y$. We will use local coordinates near the boundary of the form $(x, y_1, \ldots y_{n-1})$, where $x \geq 0$ and $x = 0$ on Y.

The goal of this talk is to describe the 'b-calculus', which is a calculus of operators on M which has a certain structure with respect to the boundary. It was formalised by Melrose although its roots go back a long way.

b-vector fields

The starting point is the set $\mathcal{V}_b(X)$ of smooth vector fields on X tangent to the boundary, called b-vector fields. In some ways, this is a more natural class of vector fields than the set of all smooth vector fields; for example, $\mathcal{V}_b$ is the set of infinitesimal generators of diffeomorphisms of X.

It turns out, somewhat surprisingly, that $\mathcal{V}_b(X)$ is the set of smooth sections of a vector bundle, the b-tangent bundle ${}^bTX$. Near the boundary, the vector fields $x\partial_x$ and ∂_{y_i}, $1 \leq i \leq n$ form a spanning set, every b-vector field can be written as a linear combination of these vector fields with smooth coefficients. In particular note that $x\partial_x$ is nonzero as a b-vector field, even at the boundary.

b-metrics

A b-metric is a nondegenerate metric on the b-tangent space. A typical example of a b-metric is

$$g = \frac{dx^2}{x^2} + h(x, y, dy), \quad h(0, y, dy) \text{ is a metric on } Y.$$

In this metric, $r = \log x$ has $|dr| \sim 1$ near the boundary. Hence, the boundary is at geometric infinity. Geometrically, the manifold has an (asymptotically) *cylindrical end* (the case where h is independent of x for small x is an exact cylindrical end).

It follows that the Laplacian with respect to such a metric is essentially self-adjoint — no boundary conditions are necessary! On the other hand, the Laplacian now has continuous spectrum.

Differential operators

To define the calculus of b-pseudodifferential operators, we first look at b-differential operators. By definition, this is the class of differential operators on X formed by taking products of b-vector fields times smooth coefficients. For example, a b-Laplacian would look something like

$$(xD_x)^2 + \sum_{i=1}^{n-1} D_{y_i}^2.$$

Consider the *kernels* (in the sense of distribution kernel, not null space!) of such operators. This is a distribution on X^2. All differential operators have kernel supported on the diagonal; they are sums of derivatives of delta functions, multiplied by smooth coefficients. To get uniformity at the boundary, it is necessary to *blow up* the corner of X^2. This space is called X_b^2.

Pseudodifferential operators

The class of b-pseudodifferential operators of order m, denoted ${}^b\Psi^m(X)$, is by definition the class of operators with kernel

(i) conormal to the diagonal of X_b^2 *uniformly* down to the boundary,

(ii) rapidly decreasing at the boundary faces lb, rb of X_b^2.

This means that locally near the boundary, the kernel of $A \in {}^b\Psi^m(X)$ has the form

$$\int e^{i\left((x'/x-1)\tau+(y'-y)\cdot\eta\right)} a(x,y,\tau,\eta)\, d\tau\, d\eta$$

where a is a symbol of order m in (τ, η).

Symbol maps

Part of the point of a calculus of pseudodifferential operators is to have useful algebraic properties. In this case, the class of operators is a graded algebra,

$$ {}^b\Psi^m(X) \cdot {}^b\Psi^l(X) \subset {}^b\Psi^{m+l}(X) $$

and there are symbol maps capturing this to some extent. The first symbol map is the principal symbol,

$$ A \mapsto \sigma^m(A) = a(x, y, \tau, \eta) \in S^m/S^{m-1}. $$

This is multiplicative:

$$ \sigma^m(A)\sigma^l(B) = \sigma^{m+l}(AB). $$

Normal operator

More interesting is the normal operator. This can be presented in two different ways. What we do first is restrict the kernel of A to bf. This gives us a function of y, y' and x'/x. Changing variable to $\log(x'/x) = \log x' - \log x = r' - r$ gives us a function of $y, y', r' - r$. This can be regarded as a convolution kernel on the cylinder $Y \times \mathbb{R}$. That is, it acts on functions on $Y \times \mathbb{R}$ via the prescription

$$f(y,r) \mapsto \int_{Y \times \mathbb{R}} N(A)(y, y', r - r') f(y', r') dy dr'.$$

Since it is translation invariant in the r variable, it makes sense to Fourier transform in this variable. Hence we define

$$I(A, \lambda)(y, y') = \int_{\mathbb{R}} e^{-is\lambda} N(A)(y, y', s)\, ds.$$

For each $\lambda \in \mathbb{R}$, $I(A, \lambda)$ operates on functions on Y. We will call this the normal, or alternatively the indicial, operator of A. This is also multiplicative:

$$I(A, \lambda) \circ I(B, \lambda) = I(AB, \lambda).$$

Fredholm properties

In index theory, we need to know when a given operator is Fredholm. First we determine when an operator is compact.

Lemma 1. *An operator* $A \in \Psi^m(X)$, $m < 0$, *is compact on* $L^2(X)$ *iff* $I(A, \lambda) \equiv 0$.

This follows from the fact that a convolution kernel is a compact operator iff it vanishes identically.

An operator A is Fredholm iff it has an approximate inverse B such that $AB - \mathrm{Id}$ is compact. It follows from the algebraic properties that we need

$$\sigma(B) = \sigma(A)^{-1}, \quad I(B, \lambda) = I(A, \lambda)^{-1}.$$

That is, A is Fredholm if its principal symbol is invertible (i.e., A is b-elliptic) and if its normal operator is invertible for all λ.

NB: an approximate inverse of $A \in {}^b\Psi^m(X)$ will generally not be in ${}^b\Psi^{-m}(X)$, but in a slightly larger class of operators where we allow the kernel to have nontrivial expansions at lb and rb.

Traces

To get numerical invariants from operators, we generally need to take a trace at some point. Here the situation is not so good; typically elements even of $\Psi^{-\infty}(X)$ are not trace class. In fact, from the discussion on compactness it follows that $I(A,\lambda)$ must vanish identically for A to be trace class. In that case, the trace can be computed by the usual formula:

$$\operatorname{tr} A = \int_X A(x,y,x,y)\frac{dx}{x}dy.$$

However, we can always, for any $A \in \Psi^{-\infty}(X)$, define the b-trace, given by

$$\begin{aligned}\text{b-tr}\, A = \lim_{\epsilon\to 0}\Big(\int_{x\geq\epsilon} A(x,y,x,y)\frac{dx}{x}dy \\ + \log\epsilon \int_Y A(0,y,0,y)dy\Big).\end{aligned}$$

Here we are doing something fairly mindless, namely just cancelling off the divergence of the integral 'by hand'. In fact this b-trace turns out to be quite useful.

The b-trace has the properties:

- If A is trace class, then $\text{b-tr}\, A = \text{tr}\, A$.
- The b-trace of a commutator $[A, B]$ is not zero, in general. However, there is a nice formula for the b-trace which is expressed entirely in terms of the boundary, where 'things went wrong':

$$\text{b-tr}[A, B] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left((\frac{d}{d\lambda} I(A, \lambda)) I(B, \lambda) \right) d\lambda.$$

Index formula for Dirac operators

To see where this can get you, consider a Dirac operator $\eth_+$ on an even dimensional manifold X with boundary. We assume that the induced Dirac operator $\eth_b$ on the boundary is invertible, which implies that $\eth$ is Fredholm as a map $H^1_b(X) \to L^2(X)$.

For a manifold M with no boundary, we can compute the index, in principle, using the McKean-Singer formula:

$$\begin{aligned} &\frac{d}{dt}\Big(\operatorname{tr} e^{-t\eth_-\eth_+} - \operatorname{tr} e^{-t\eth_+\eth_-}\Big) \\ &= -\Big(\operatorname{tr} \eth_-\eth_+ e^{-t\eth_-\eth_+} - \operatorname{tr} \eth_+\eth_- e^{-t\eth_+\eth_-}\Big) \\ &= -\Big(\operatorname{tr} \eth_-\eth_+ e^{-t\eth_-\eth_+} - \operatorname{tr} \eth_- e^{-t\eth_+\eth_-}\eth_+\Big) \\ &= 0, \end{aligned}$$

because $\eth_+ e^{-t\eth_-\eth_+} = e^{-t\eth_+\eth_-}\eth_+$ (this is clear for bounded operators, by Taylor-expanding the exponential).

Sending $t \to \infty$,

$$\operatorname{tr} e^{-t\eth_-\eth_+} \to \dim \operatorname{null} \eth_+,$$

and similarly

$$\operatorname{tr} e^{-t\eth_+\eth_-} \to \dim \operatorname{null} \eth_-,$$

so the expression above is equal to the index of $\eth_+$ for every value of t. If we now send t to zero, we get a local expression (in terms of the curvature function on M) for the index.

Index theory — manifolds with boundary

Now we run the same argument on manifolds with boundary, replacing the trace with the b-trace and see what happens. So consider

$$\frac{d}{dt}\Big(\,\text{b-tr}\, e^{-t\eth_-\eth_+} - \text{b-tr}\, e^{-t\eth_+\eth_-}\Big).$$

As $t \to 0$ we still get a local expansion. As $t \to \infty$, the kernel $e^{-t\eth_-\eth_+}$ approaches the projection onto the null space exponentially, hence

$$\begin{aligned}\text{b-tr}\, e^{-t\eth_-\eth_+} &\to \text{b-tr}\,\Pi_0(\eth_-\eth_+)\\ &= \operatorname{tr}\Pi_0(\eth_-\eth_+) = \dim\operatorname{null}\eth_+.\end{aligned}$$

As for the time derivative, that will no longer be zero, because we used the fact that the trace of a commutator is zero. In our case, we use the formula for the b-trace of a commutator and find that

$$\begin{aligned}&\frac{d}{dt}\Big(\,\text{b-tr}\, e^{-t\eth_-\eth_+} - \text{b-tr}\, e^{-t\eth_+\eth_-}\Big)\\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty}\big(\frac{d}{d\lambda}I(\eth_+,\lambda)\big)I(\eth_- e^{-t\eth_-\eth_+},\lambda)\,d\lambda\\ &= -\frac{1}{2\sqrt{\pi}}t^{-1/2}\operatorname{tr}\eth_b e^{-t\eth_b^2}.\end{aligned}$$

We end up with the Atiyah-Patodi-Singer formula for the index of a Dirac operator, in the context of the b-calculus:

$$\operatorname{ind} \eth_+ = \int_X \hat{A} - \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{tr} \eth_b e^{-t\eth_b^2} \, dt.$$

This is very close to the standard form of the APS Theorem using APS boundary conditions, since

• APS boundary conditions mimic a cylindrical end attached to the boundary, and

• the A-hat integrand vanishes (pointwise) on a cylinder $\partial X \times \mathbb{R}$.