Eigenvalues and eigenfunctions of the Laplacian

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The setting

In this talk I will consider the Laplace operator, Δ, on various geometric spaces M. Here, M will be either a bounded Euclidean domain, or a compact Riemannian manifold with metric g, with smooth (or perhaps only piecewise smooth) boundary. The disc, B^2, the flat torus, T^2, and the round sphere, S^2, will be important two-dimensional examples. In either case, we can define

• an inner product on functions on M by

$$\langle f, g\rangle = \int_M f(x)\overline{g(x)}d\mu.$$

Here, μ is the natural measure (Lebesgue measure in the case of a Euclidean domain, Riemannian measure otherwise).

• the lengths of vectors on M, in particular the gradient vector ∇f.

The Laplacian applied to a function f, Δf, is defined by the condition that

$$\langle \Delta f, g \rangle = \langle \nabla f, \nabla g \rangle$$

for every function g with square-integrable derivatives. If M has boundary, then we require in addition that g vanishes at the boundary. This defines the Laplacian with Dirichlet boundary conditions (f vanishing at the boundary).

On a Euclidean domain,

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

The Laplacian Δ is a self-adjoint operator on $L^2(M)$. Moreover, for bounded M, it has pure-point spectrum. In fact, there is a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \to \infty$$

and an orthonormal basis $u_1, u_2, \ldots$ of $L^2(M)$, which are eigenfunctions of Δ:

$$\Delta u_j = \lambda_j^2 u_j, \ \|u_j\|_{L^2(M)} = 1.$$

In this talk I want to discuss asymptotic properties of the sequence of eigenvalues and eigenfunctions and relate them to the geometry of (M, g).

Examples

• The interval $[0,a]$. Eigenfunctions and eigenvalues are

$$u_n = \sqrt{\frac{2}{a}} \sin \frac{\pi n x}{a}, \quad \lambda_n = \frac{\pi n}{a}.$$

• The torus T^2_π. Eigenfunctions and eigenvalues are

$$u = \frac{1}{\pi} e^{ilx} e^{imy}, \quad \lambda = \sqrt{l^2 + m^2}.$$

• The sphere S^2. Eigenfunctions and eigenvalues are

$$u = Ce^{im\phi} P^l_m(\cos\theta), -l \le m \le l,$$

$$\lambda = \sqrt{l(l+1)} \text{ with multiplicity } 2l+1.$$

Here P^l_m is a special function; particular cases are

$$P^l_l(\cos\theta) = (\sin\theta)^{l+1}, \quad P^l_0(\cos\theta) = \ldots$$

Finally, an example which does not quite fit into the framework above, but is very illuminating. Consider the 'semiclassical' problem

$$(h^2\Delta + x^2 - 1)u(x,h) = 0, \quad x \in \mathbb{R},$$

For which h is there a nontrivial L^2 solution? Then $\lambda^2 = h^{-2}$ is analogous to an eigenvalue. The answer is that $h_n^{-1} = \lambda_n = 2n+1$, and

$$u_n = c_n h_n(\sqrt{2n+1}\, x)e^{-(2n+1)x^2/2},$$

where c_n is a normalizing factor, and h_n is the nth Hermite polynomial.

Asymptotic distribution of eigenvalues

How are the eigenvalues distributed, asymptotically? — ie, what are the asymptotics of

$$N(\lambda) = \#\{j \mid \lambda_j \leq \lambda\} \text{ as } \lambda \to \infty?$$

$N(\lambda)$ is a nondecreasing, integer valued function which counts the number of eigenvalues less than a given value.

Weyl's Law:

$$N(\lambda) = C_n \operatorname{vol}(M)\lambda^n + O(\lambda^{n-1}),\ n = \dim M.$$

Classical interpretation: The classical hamiltonian corresponding to the quantum hamiltonian Δ is the symbol of Δ, namely the function on phase space given by $p(x, \xi) = |\xi|^2$. Here phase space is $M \times \mathbb{R}^n$ for a Euclidean domain M, or more generally the cotangent bundle T^*M, and $|\xi|^2$ is the length with respect to the metric, thus

$$p(x,\xi) = \sum_{i,j} g^{ij}(x)\xi_i\xi_j$$

for a metric. The *classically allowed region* is the part of phase space where the hamiltonian is no bigger than λ^2. This is

$$\{(x,\xi) \mid |\xi| \leq \lambda\}, \text{ volume } = c\,\mathrm{vol}(M)\lambda^n.$$

Then the physical interpretation is that each quantum state (eigenfunction) occupies a fixed volume of phase space - cf. uncertainty principle.

Weyl's Law can be proved in various ways. One is to study an auxiliary differential equation, such as the heat equation

$$\frac{\partial u}{\partial t} = -\Delta u, \quad u(x,0) = f(x),$$

where u is a function of $x \in M$ and time t. An example of a solution to this equation is

$$e^{-\lambda_j^2 t} u_j(x),$$

for any eigenpair (λ_j, u_j). This PDE has a fundamental solution $K(x,y,t)$ and spectral theory shows that

$$\int_M K(x,x,t)d\mu = \sum_j e^{-t\lambda_j^2}.$$

On the other hand, PDE theory shows that (on a Euclidean domain, and away from the boundary)

$$K(x,x,t) \sim (4\pi t)^{-n/2}, \quad t \to 0.$$

Now, very roughly, $e^{-t\lambda_j^2}$ is approximately 1 if $t < \lambda_j^{-2}$ and is about zero if $t > \lambda_j^{-2}$. So we see that the number of λ_j such that $t < \lambda_j^{-2}$ is about $(4\pi t)^{-n/2}\,\mathrm{vol}(M)$.

This argument can be made precise, but gives an inferior error term. To get the error term stated above, you need to use the wave equation instead of the heat equation, and some Fourier analysis.

Error term

The error term controls all sorts of quantities, for example, the multiplicity of an eigenvalue. To see this, note that the multiplicity of λ is given by

$$\lim_{\epsilon \to 0} N(\lambda + \epsilon) - N(\lambda) = O(\lambda^{n-1}).$$

Surprisingly, this result is sharp: for the sphere S^n, the eigenvalues are at $\lambda^2 = l(l + n - 1)$, with multiplicity $\sim l^{n-1}$. What is it about the sphere that gives rise to high multiplicity?

Theorem 1. *(Duistermaat-Guillemin, Ivrii) Suppose that the set of periodic geodesics (billiard trajectories) on M has measure zero in T^*M. Then*

$$N(\lambda) = C_n \operatorname{vol}(M)\lambda^n + C'_n \operatorname{vol}(\partial M)\lambda^{n-1} + o(\lambda^{n-1}).$$

Improving the error estimate is difficult. For example, for the torus, the error term is

$$E(r) = \#\{(m,n) \mid m^2 + n^2 \leq r^2\} - \pi r^2$$

is the number of integer lattice points inside a circle of radius r, less the area of the circle. Proving $E(r) = O(r)$ is straightforward. It is thought that

$$E(r) = O(r^{1/2+\epsilon}) \text{ for every } \epsilon > 0.$$

However, this is a very difficult problem. The best result so far is that (Huxley)

$$E(r) = O(r^{46/73}(\log r)^{315/146}).$$

The Riemann Hypothesis has an equivalent formulation in terms of the counting problem for *primitive* lattice points.

Quantum limits

Given a subsequence $u_{k_1}, u_{k_2}, \ldots$ of eigenfunctions, $k_1 < k_2 < \ldots$, when does the sequence of probability measures

$$|u_{k_1}|^2, |u_{k_2}|^2, \ldots$$

converge?

First we must choose a reasonable notion of convergence. One dimensional examples show that the right notion is weak* convergence of measures.

For example, the sequence of eigenfunctions for the interval $[0, a]$ converge weakly to $a^{-1}\chi_{[0,a]}$, while the sequence of eigenfunctions of the semiclassical harmonic oscillator converges weakly to the measure

$$\frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}\,\chi_{[-1,1]}.$$

More subtly, a sequence of eigenfunctions may give rise to a measure on phase space, T^*M.

Before, we were effectively looking at 'expectation values'

$$\langle au_{k_j}, u_{k_j} \rangle, \quad a \in C(M).$$

We can write a as an integral operator

$$a(x)\delta(x-y) = (2\pi h)^{-n} \int e^{i(x-y)\cdot\xi/h} a(x)\, d\xi.$$

Now we consider more general integral operators, depending on a parameter $h > 0$

$$A_h(x, y) = (2\pi h)^{-n} \int e^{i(x-y)\cdot\xi/h} a(x, \xi, h)\, d\xi.$$

Here h is a scale parameter; the length scale is $O(h)$ and consequently the frequency scale is $O(h^{-1})$. The function a 'microlocalizes'.

Given a subsequence of eigenfunctions, we ask whether

$$\langle A_{\lambda_{k_j}^{-1}} u_{k_j}, u_{k_j} \rangle$$

converges as $j \to \infty$, for each operator A_h.

Fact: if the limit does exist, then it depends only on the function $a(x, \xi, 0)$ restricted to the 'propagating set'

$$\mathcal{P} = \{(x, \xi) \mid |\xi|^2 = 1\}$$

and

$$\left| \lim_j \langle A_{\lambda_{k_j}^{-1}} u_{k_j}, u_{k_j} \rangle \right| \leq \sup_{(x,\xi) \in \mathcal{P}} |a(x, \xi, 0)|.$$

That implies that the sequence of eigenfunctions determines a *measure* on $\mathcal{P}$. Such a measure is called a *quantum limit* of Δ.

Examples. Unit interval, harmonic oscillator, torus, equator of a sphere.

Which measures can occur as quantum limits? (There must exist at least one quantum limit, by weak* compactness of the unit ball in the space of bounded Borel measures on $\mathcal{P}$.)

Theorem 2. *(Hörmander) Any quantum limit is invariant under geodesic flow.*

This places some limitations on the sort of measures that can turn up as quantum limits, but it still leaves open a lot of possibilities. The extreme possibilities are Liouville measure on $\mathcal{P}$, or measures supported by periodic geodesics.

Quantum ergodicity

One result concerns ergodic spaces. By definition, this means that the propagating set $\mathcal{P}$ has positive measure invariant subspaces under geodesic flow (except for subspaces of full measure). Then, almost all geodesics are distributed evenly over $\mathcal{P}$: if the geodesic is $\gamma(s) \subset \mathcal{P}$, parametrized by arclength, and E is a measurable subset of $\mathcal{P}$, then

$$\lim_{T\to\infty} \frac{1}{2T}\Big|\{s \in [-T,T] \mid \gamma(s) \in E\}\Big| = \frac{|E|}{|\mathcal{P}|}$$

for almost all γ.

Theorem 3. *(Schnirelman, Zelditch, Colin de Verdiere) If the geodesic flow on $\mathcal{P}$ is ergodic then there is a subsequence of eigenfunctions of density one whose quantum limit is Liouville measure (ie, M is quantum ergodic).*

The opposite extreme, concentration at a single periodic geodesic, can also happen under favourable conditions.

Theorem 4. *(Babich/Lazutkin/Guillemin) Let γ be a stable periodic geodesic. Then there is a sequence of eigenfunctions with quantum limit supported on γ.*

Another case is when the classical hamiltonian p on M is completely integrable. This means that T^*M is foliated by invariant tori, and the geodesic flow is quasiperiodic on each torus.

Theorem 5. *Suppose that (M, g) is completely integrable. Then each invariant torus supports a quantum limit.*

One can also look at the boundary traces (normal derivative at the boundary, for Dirichlet eigenfunctions, value at the boundary otherwise) of eigenfunctions.

Theorem 6. *(H. - Zelditch, Burq) Let M be an ergodic Riemannian manifold with piecewise smooth boundary. For each of a large class of boundary conditions, there is a measure supported on the unit ball bundle $B^*(\partial M)$ which is the quantum limit of the boundary traces of a density one sequence of eigenfunctions.*

Quantum unique ergodicity

One question, which has assumed the status of an major unsolved problem in the area, is whether classically ergodic domains (of dimension ≥ 2) are quantum unique ergodic. That is, is Liouville measure the only quantum limit for such systems? The general feeling is as follows: compact manifolds of negative curvature are probably QUE, while the Bunimovich stadium is probably non-QUE. However, both of these expectations could easily be wrong. There isn't a single ergodic example whose QUE status is known! (but note Rudnick-Sarnak result for certain arithmetic surfaces).

L^∞ bounds on eigenfunctions

It is natural to ask about pointwise bounds on eigenfunctions. How big can

$$\|u_j\|_{L^\infty} = \sup_x |u_j(x)|$$

be, in terms of the eigenvalue λ_j?

The best that can be said, without making geometric assumptions, is

$$\|u_j\|_{L^\infty} \leq C\lambda_j^{(n-1)/2}.$$

This is sharp for the sphere S^n. On S^2, the spherical harmonics Y_l^0 accumulate at the north and south poles, with size $\sim \lambda^{1/2}$ there. The theorem of Duistermaat-Guillemin and Ivrii gives an estimate

$$\|u_j\|_{L^\infty} = o(\lambda_j^{(n-1)/2})$$

if the set of periodic geodesics has measure 0.

Theorem 7. *(Sogge-Zelditch) If a compact manifold M without boundary has maximal eigenfunction growth, then there is a point $x \in M$ with a positive measure of geodesics that return to x. As a corollary, if (M, g) is two dimensional and real analytic, then M is topologically S^2.*

There is also a result in the other direction. A manifold has minimum eigenfunction growth if the $\|u_j\|_{L^\infty}$ are uniformly bounded. Notice that a torus has this property, since the eigenfunctions are exponentials.

Theorem 8. *(Toth, Zelditch) Suppose (M, g) is a compact Riemannian manifold without boundary which is completely integrable and satisfies a finite complexity condition. Then if the eigenfunctions are uniformly bounded in $L^\infty(M)$, (M, g) is a flat torus.*

Bounds on boundary traces

One can ask about L^p bounds on the boundary traces of eigenfunctions (ie, the normal derivative for Dirichlet eigenfunctions, the restriction of the eigenfunction for Neumann eigenfunctions)

Theorem 9. *(Bardot-Lebeau-Rauch, H.-Tao) If the manifold M has no trapped geodesics, then for Dirichlet eigenfunctions*

$$c\lambda_j \leq \|\partial_\nu u_j\|_{L^2(\partial M)} \leq C\lambda_j.$$

The result for Neumann is as follows; surprisingly it is sharp for the ball.

Theorem 10. *(Tataru) For Neumann eigenfunctions*

$$\|u_j\|_{L^2(\partial M)} \leq \lambda_j^{1/3}.$$