THE SCHRODINGER PROPAGATOR FOR SCATTERING METRICS

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Schrödinger equation for free particle:
\[
\left( i^{-1} \partial_t + \frac{1}{2} \Delta \right) \psi = 0 \quad \psi|_{t=0} = \psi_0.
\]
On curved space: \( \Delta = \Delta_g = \nabla^*_g \nabla \geq 0 \).

For example, on flat \( \mathbb{R}^n \), the fundamental solution is
\[
U(t, w, x) = (2\pi it)^{-n/2} e^{i|x-w|^2/2t}.
\]
- infinite speed of propagation
- lack of decay (unitarity)

Fix \( t > 0, x \). Then
\[
u(t, x, w) \text{ is smooth in } w, \text{ while}
\]
\[
u(0, x, w) = \delta(w - x) : \text{singularity disappears instantly.}
\]

Initial data \( \psi_0 = e^{i(-\lambda x^2/2 + \xi \cdot x)} : \psi_0 \in \mathcal{C}\infty \)
but
\[
\psi(t, w)|_{t=\lambda^{-1}} = C\delta(w + \xi/\lambda).
\]
Singularity appears!
What happens on curved space?

Kapitanski-Safarov (’96):
If no trapped geodesics, \( \psi_0 \in E' \Rightarrow \psi(t) \in C^\infty \) for all \( t > 0 \).

(’98): Parametrix modulo \( C^\infty(\mathbb{R}^n) \), but no control at \( \infty \).

Craig-Kappeler-Strauss (’95):
Regularity of the solution at all times, and in certain directions, under assumption of regularity of \( \psi_0 \) along all geodesics lying inside a given spatial cone near infinity.

Wunsch (’98):
Regularity along certain nontrapped geodesics at particular \( t > 0 \) described by “quadratic-scattering wavefront set” of \( \psi_0 \):
specifies Directions and times of singularities (location still still mysterious).
Robbiano-Zuily (’02):
Analogue of Wunsch’s result in analytic category.

Burq-Gérard-Tzvetkov (’01), Staffilani-Tataru (’02): Strichartz estimates.

Contrast with well-developed theory for wave equation. Consider

$$(\partial_t - i\sqrt{\Delta})u = 0, \quad u(0) = u_0$$
on a compact, boundaryless Riemannian manifold $M$. Solution is $u(t) = e^{it\sqrt{\Delta}}u_0$. Let $\Phi_t$ be geodesic flow on $S^*X$ at time $t$.

**Theorem** (Hörmander)

1. $e^{it\sqrt{\Delta}}$ is a *Fourier integral operator* which quantizes the contact transformation $\Phi_t$.
2. $(x, \hat{\xi}) \in \text{WF} u_0$ iff $\Phi_t(x, \hat{\xi}) \in \text{WF} u(t)$.

- $\Phi_t$ is a contact transformation of the contact manifold $S^*X$ with contact one-form $\hat{\xi} dx$.
- Singularities travel with unit speed along geodesics, and are neither created nor destroyed (time reversibility).
- Statement 2 follows immediately from statement 1.
Back to Schrödinger

Goal: construct parametrix, describe regularity of $\psi(t)$.

Specific questions:
(1) When and where can singularities appear in $\psi(t)$? (Describe in terms of initial data.)
(2) Where do singularities of initial data in $\mathcal{E}'$ disappear to?
(3) What is the structure of the fundamental solution with initial pole at $x$?

Questions (1) and (2) are dual. A strong enough answer to (3) will address both.
**General geometric setup:** $(X, g)$ a Riemannian manifold with ends that look asymptotically like the large ends of cones $(1, \infty) \times Y$: ( = ‘manifold with scattering metric’ as defined by Melrose):

$$g = dr^2 + r^2 h(r^{-1}, y, dy)$$

$$h \in C^\infty, \ h_0 \equiv h(0, y, dy) \text{ a metric on } Y.$$

**Key example:** $X =$ asymptotically Euclidean space; $r = |x|, y = \theta = x/|x| \in S^{n-1}$:

$$g = (1 + \frac{2m}{r})dr^2 + r^2 d\theta^2 + O(r^{-2})(dr, rd\theta), \ r \to \infty$$

(We stick with this example for remainder of talk.)

**Crucial geometric assumption:**

**no trapped geodesics**

(Or, stay microlocally away from trapping region.)
Hamiltonian:
\[ H \equiv \frac{1}{2} \Delta g + V(x) \]

with \( V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) having asymptotic expansion:
\[ V(x) \sim \frac{c}{r} + r^{-2}V_{-2}(\theta) + r^{-3}V_{-3}(\theta) + \ldots \]
\[ \in r^{-1}C^\infty(r^{-1}, \theta), \text{ for } r \geq r_0 > 0. \]

• Newtonian gravity (the \( 1/r \) term in the potential) and Einsteinian gravity (the \( dr/r \) term in the metric) are both OK.

Schrödinger equation now reads
\[
\boxed{(i^{-1}\partial_t + H)\psi = 0.}
\]

and we are interested in the kernel of the fundamental solution, \( e^{-itH} \).
Re-examine *Euclidean* fundamental solution (with $V = 0$):

Let $r = |w|$, $\theta = w/|w|$. Then

$$e^{-itH} \delta_x = (2\pi it)^{-n/2} e^{i|x-w|^2/2t}$$

$$= e^{ir^2/2t} \left( ae^{-i(rx \cdot \theta - |x|^2/2)/t} \right)$$

with $a = (2\pi t)^{-n/2}$.

We start at $t = 0$ with $\delta_x$. Study later behavior of solution in $r, \theta$ variables ($x, t > 0$, fixed):

- The $e^{ir^2/2t}$ term is independent of $x$: loses all information about location of initial singularity.
- The $e^{-i(rx \cdot \theta - |x|^2/2)/t}$ term retains information about location of initial pole in its oscillation as $r \to \infty$.
- The time $t$ appears in the phases in a very simple way.
Use this form as ansatz in more general geometric setting: divide by the explicit quadratic oscillatory factor $e^{ir^2/2t}$ and try to construct the resulting kernel, which is hopefully only linearly oscillatory.

**Theorem.** Let $\chi \in C^\infty_c(\mathbb{R}^n)$. The fundamental solution is of the form

$$e^{-itH}\chi = e^{ir^2/2t}W_t\chi,$$

where the kernel of $W_t$ is a *scattering fibered Legendrian* (Melrose-Zworski, H.-Vasy).

- Inserting the function $\chi$ means that we only consider the asymptotics as $|w| \to \infty$, keeping $x$ in a fixed (but arbitrary) compact set.

On $\mathbb{R}_t \times \mathbb{R}^{n_r}_r \times \mathbb{R}^{n_\theta}_\theta \times \mathbb{R}^n_x$, $W$ is a finite sum of terms of the form

$$(0.1)\quad t^{-n/2-k/2} \int_{U \subseteq \mathbb{R}^k} a(t, r^{-1}, \theta, x, v)e^{i\phi(r^{-1}, \theta, x, v)r/t} \, dv.$$
We can state a slightly weaker version of the theorem more easily by composing with the Fourier transform \( \mathcal{F} \):

Let \( W_t = e^{-ir^2/2t}e^{-itH} \) for fixed \( t > 0 \). Then \( \mathcal{F} \circ W_t \) is a Fourier integral operator.

To analyze \( W_t \) further we recall the definition of the scattering wavefront set, which in \( \mathbb{R}^n \) can be specified in terms of the usual wavefront set and the Fourier transform.

Let \( S^{n-1}_\infty \) denote the “sphere at infinity” of our asymptotically Euclidian space.

Can identify

\[
S^* \mathbb{R}^n \equiv \mathbb{R}^n \times S^{n-1}, \\
T^*_{S^{n-1}_\infty} \mathbb{R}^n \equiv S^{n-1} \times \mathbb{R}^n.
\]

Hence exchanging coordinates

\[
(\theta, \zeta) \rightarrow (\zeta, \theta)
\]

gives diffeomorphism between these spaces (and gives \( T^*_{S^{n-1}_\infty} \mathbb{R}^n \) a contact structure).
**Definition.** The *scattering wavefront set* of a distribution $u$ is the subset of $T_{S_{\infty}^{n-1}}^* \mathbb{R}^n$ defined by

$$(\theta, \zeta) \in WF_{sc}(u) \text{ iff } (\zeta, \theta) \in WF(\mathcal{F}u).$$

(Definition originates with Melrose in more general setting (scattering metrics).)

$WF_{sc}$ measures linear oscillation near infinity. For example, let $u(x) = e^{i\alpha \cdot x}$. Then

$$WF_{sc}u = \{(\theta, \alpha) : \theta \in S_{\infty}^{n-1}\}.$$  

Hence $e^{-ir^2/2t}e^{-itH}$ is a “scattering FIO” interchanging scattering wavefront set and ordinary wavefront set.
**Question:** what is the canonical relation of $W_t = e^{-ir^2/2t} e^{-itH}$?

Let $(x, \hat{\xi}) \in S^*\mathbb{R}^n$. Let $\gamma(t)$ be geodesic with $\gamma(0) = x$, $\gamma'(0) = \hat{\xi}$.

Define $\Phi : S^*\mathbb{R}^n \to T^* S^*_\infty$ by

$$\Phi(x, \hat{\xi}) = (\theta, \lambda \theta + \mu) \text{ with } \mu \perp \theta$$

given by

$$\theta = \lim_{t \to \infty} \frac{\gamma(t)}{|\gamma(t)|} \in S^{n-1}_\infty$$

$$\lambda = \lim_{t \to \infty} t - |\gamma(t)|$$

$$\mu = \lim_{t \to \infty} |\gamma| \left( \frac{\gamma(t)}{|\gamma(t)|} - \theta \right)$$

Thus

- $\theta$ is asymptotic direction;
- $\lambda$ is “sojourn time” (cf. Guillemin) that a particle spends in finite region before heading out to $\infty$ (finite by assumption);
- $\mu$ measures angle of approach to $S^{n-1}_\infty$. 
Proposition. $\Phi$ is a contact transformation from $S^*\mathbb{R}^n$ to $T^*_{S^\infty_{n-1}}\mathbb{R}^n$.

**The canonical relation for $W$:**

The canonical relation parametrized by $W_t = e^{-ir^2/2t}e^{-itH}$ is $t^{-1}\Phi$ (scaling acts in fiber variable).

**Special case:** $x \in \mathbb{R}^n$, $\theta \in S^\infty_{n-1}$, and there exists a unique geodesic $\gamma(t)$ from $x$ to $\theta$ (non-degenerate case). Then locally

$$W = ae^{irS(x,\theta)/t}$$

(no integral required), where

$$S(x, \theta) = \lim_{t \to \infty} t - |\gamma(t)|.$$  

“sojourn time.”
Euclidean example once more:

\[ e^{-ir^2/2t} e^{-itH} \hat{\delta}_x = ae^{i(-x \cdot \theta + O(r^{-1}))} r/t. \]

Sojourn time in \( \mathbb{R}^n \) for line through \( x \) in direction \( \theta \):

\[ S(x, \theta) = \lim_{t \to \infty} t - |x + t\theta| = -x \cdot \theta, \]

as appears in phase!
**Egorov Theorem.** Let $A$ be a properly supported, zeroth order pseudo on $\mathbb{R}^n$. Then

$$\tilde{A} = \mathcal{W}_t AW_t^*$$

is a zeroth order scattering pseudodifferential operator, and

$$\sigma_{sc}(\tilde{A})(\Phi(q)) = \sigma(A)(q).$$

**Propagation Theorem.** Let $\psi(t) = e^{-itH}\psi_0$. Fix a $t \neq 0$. Then

$$(x, \hat{\xi}) \in \text{WF} \psi(t)$$

iff

$$-\frac{1}{t}\Phi(x, -\hat{\xi}) \in \text{WF}_{sc}(e^{ir^2/(2t^2)}\psi_0).$$

Thus, we have a characterization of the singularities at nonzero time $t$ in terms of the asymptotic behaviour of the initial data. Previous propagation results are immediate consequences.
Some words on the proof

A parametrix for $e^{-itH}$ is constructed as a Legendrian distribution, starting at the diagonal near $t = 0$. Here it takes the form

$$U(w, x, t) = e^{i\Psi(w, x)/t} a(t, w, x), \quad a \text{ smooth},$$

with $\Psi(w, x)$ equal to $d(w, x)^2/2$. The function $\Psi$ determines a Legendrian submanifold of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times \mathbb{R}$, namely

$$L = \{(w, \zeta, x, \xi, \tau) \mid \zeta = d_w \Psi, \xi = d_x \Psi, \tau = \Psi\}$$

which is Legendrian with respect to the contact form $\zeta \cdot dw + \xi \cdot dx - d\tau$. This Legendrian becomes non-projectable outside the injectivity radius, meaning $(w, x)$ are no longer coordinates on it, but it remains perfectly smooth. It may be defined by

$$(w, \zeta) = \exp_{sg/2}(x, \xi), \tau = s^2/2, \quad s \in (0, \infty).$$
We investigate the behaviour of $L$ as $s \to \infty$. We see that $|\zeta|$ and $|\xi|$ grow linearly, and $\tau$ quadratically, as $s \to \infty$. Moreover, the form of the metric is such that $r = |w| \sim s$ as $s \to \infty$, provided the metric is nontrapping. So it makes sense to introduce scaled variables

$$\bar{\zeta} = \rho \zeta, \quad \bar{\xi} = \rho \xi, \quad \kappa = \rho^2 \tau,$$

where $\rho \equiv r^{-1} \to 0$. We also write

$$\bar{\zeta} = \bar{\nu} \hat{w} + \bar{\mu} \quad \text{where} \quad \bar{\mu} \perp \hat{w}.$$

If we do this, then we find that along every geodesic,

$$\bar{\nu} \to -1, \quad \bar{\mu} \to 0, \quad \kappa \to -1/2, \quad \bar{\xi} \to \xi_0,$$

so the submanifold $L$ is ‘bounded’ in terms of this scaling.
However, although bounded, $L$ isn’t smooth at $\rho = 0$; instead it has a conic singularity there. But this can be resolved by blowing up the set
$$\{\rho = 0, \nu = -1, \mu = 0\}.$$ 
This blowup desingularizes the submanifold $L$, which now meets the boundary of the blown-up space transversally. Moreover, the boundary of the blown-up space is a copy of $T^*_{S^{n-1}_\infty} \mathbb{R}^n$. So, we can start at any initial point $(x, \hat{\xi})$ and travel along the corresponding geodesic, eventually arriving at a point on the blown-up face which can identified with a point of $T^*_{S^{n-1}_\infty} \mathbb{R}^n$. This, by definition, is $\Phi((x, \hat{\xi}))$. Moreover, the symplectic nature of the construction implies that $\Phi$ is a contact transformation. The fact that $\Phi$ is contact in turn allows the application of the theory of Legendre distributions on manifolds with corners.
The operation of blowing up
\[ \{ \rho = 0, \bar{\nu} = -1, \bar{\mu} = 0 \} \]
is the exact geometric analogue of removing the factor \( e^{ir^2/2t} \) from the propagator. The boundary face created by blowup is ‘one order better in \( \rho \)’, and corresponds analytically to linear, rather than quadratic, oscillations. Indeed, parametrizing the submanifold \( L \) near the blowup gives us the phase function \( \phi \) is in (0.1). The theory of fibred Legendre distributions then tells us that \( \phi \) is independent of \( t \) (as in the free case) and gives us the very simple behaviour in \( t \) of singularity propagation for \( e^{-itH} \).