LECTURE 3: RIEMANN'S GEOMETRY

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1. Summary of Lecture 2.

- We showed that the Lorentzian geometry of Minkowski space led directly and straightforwardly to the well-known consequences of special relativity.
- We saw how force, power, energy and momentum no longer define invariant quantities in relativity and must necessarily be unified into 4-dimensional invariants 'power-force' and 'energy-momentum'. Mass, however, is an invariant quantity (it's the length of the energy-momentum vector).
- We discussed the electromagnetic force and, briefly, Maxwell's equations, which are unified beautifully into two exterior equations in the Minkowskian framework of special relativity.

"I do not believe in mathematics." - Albert Einstein[1]

"Grossmann, you must help me or else I'll go crazy!" - Albert Einstein[2]

"I assure you that with respect to the quantum, I have nothing new to say... I am now exclusively occupied with the problem of gravitation and I hope to master all difficulties with the help of a friendly mathematician here. But one thing is certain: in all my life I have labored not nearly as hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now. Compared with this problem, the original relativity is child's play." - Albert Einstein[3]

Georg Friedrich Bernhard Riemann (1826 – 1866).

[1]Einstein is purported to have said this before 1910, although I cannot find a solid source for this. Besides, it is clearly taken out of context. In any case, it is clear that, before his work with Grossmann, Einstein certainly believed that overuse of mathematics in physics was undesirable.

[2]In a letter to Marcel Grossmann August 1912.

[3]In a letter to Arnold Sommerfeld 29 October 1912.

In this lecture, we'll study the beautiful language in which the theory of general relativity is written: Riemannian geometry. The development of Riemannian geometry and general relativity are closely tied, with both mathematicians and physicists making significant contributions to both fields.

2. Housekeeping: Manifolds and Miscellany

Recall that a *manifold* M is a topological space[4] that 'looks locally like' Euclidean space. By 'locally', we mean that each point p has a neighbourhood which looks like $\mathbb{R}^n$. By 'looks like', we mean different things depending on the type of structure in question. For a *topological manifold*, 'looks like' means homeomorphic: there is a continuous map ϕ_p of the neighbourhood to $\mathbb{R}^n$ that is invertible, and the inverse is also continuous. The pair (U_p, ϕ_p) is called a *coordinate chart*. An *atlas* for the manifold is a compatible class of charts $\{(U_a, \phi_a)\}$ that cover the manifold. That is $M = \bigcup_a U_a$, and, whenever $U_a \cap U_b$ is non-empty, the *transition function* $\phi_a \circ \phi_b^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. A *differentiable manifold* is a topological manifold with an atlas whose transition functions are diffeomorphisms. That is, they are differentiable invertible maps with differentiable inverses. The atlas is called a *differentiable structure* for the manifold, and allows us to differentiate functions f at a point p by differentiating the corresponding function $f \circ \phi_p^{-1} : \mathbb{R}^n \to \mathbb{R}$ of the coordinates. The set of functions for which this is possible everywhere are called the differentiable functions. A *smooth manifold* has smooth (i.e differentiable of all orders) transition functions. From now on, manifold will mean smooth manifold, and function will mean smooth function. We denote the set of smooth functions on M by $C^\infty(M)$, which forms a ring under pointwise addition and scalar multiplication.

Aristotlean, Galilean and Minkowskian spacetime are all examples of manifolds. In each case we were able to define global coordinate charts. This is not always the case – for example, the sphere admits no global coordinate chart.

2.1. **Tangent vectors.**

The tangent plane at some point on a surface in $\mathbb{R}^3$ is the span of all tangent vectors (at the point) to smooth curves that lie in the surface and pass through the point. Phrased in this way, the idea generalises easily to manifolds: consider curves $\gamma : I :\to M$, where $I \subset \mathbb{R}$ is some interval. We say that γ is smooth if, given some (and hence any) local coordinates $\phi : U \to \mathbb{R}^n$, the curve $\phi \circ \gamma : I \to \mathbb{R}^n$ is smooth. The *tangent space* to M at p, denoted T_pM, is the set of equivalence classes of curves through p having the same tangent at p:

$$T_pM := \{\gamma' := [\gamma]_\sim\},$$

where $\gamma \sim \lambda$ if $\gamma(0) = \lambda(0) = p$ and

$$\left.\frac{d}{ds}\right|_{s=0} (\phi \circ \gamma) = \left.\frac{d}{ds}\right|_{s=0} (\phi \circ \lambda)$$

for some (and hence any) chart (U, ϕ) about p.

[4] A space where 'open set' makes sense.

The tangent vectors have a natural action on functions:

$$\gamma' f = \left.\frac{d}{ds}\right|_{s=0} (f \circ \gamma)$$

for any representative γ. This action of tangent vectors on functions identifies T_pM with the space of derivations, v, of smooth functions at p:

$$\begin{aligned} v : C^\infty(p) &\to \mathbb{R} \\ \text{such that} \quad v(af+g) &= avf + vg \\ \text{and} \quad v(fg) &= (vf)g(p) + f(p)(vg) \end{aligned}$$

for all $a \in \mathbb{R}$ and $f, g \in C^\infty(p)$ (the functions that are smooth at p).

Now, given some coordinate chart $\phi : U \to \mathbb{R}^n$, whose components we denote $x^i : U \to \mathbb{R}$, we can define derivations $\partial_i|_p \equiv \partial_{x^i}|_p$ by

$$\partial_i|_p f := \left.\frac{\partial}{\partial x^i}\right|_p f \circ \phi^{-1} ,$$

In fact[5], at each point $p \in U$, the derivations $\partial_i|_p$ form a basis for the space of derivations at p (and hence T_pM).

The *tangent bundle* of M, denoted TM, is the disjoint union of all of the tangent spaces:

$$TM := \sqcup_{p\in M} T_pM := \cup_{p\in M}\{(p,v) : v \in T_pM\} .$$

So its elements are pairs (p, v) of points $p \in M$ and tangent vectors $v \in T_pM$. There is a natural projection map

$$\begin{aligned} \pi : TM &\to M \\ (p,v) &\mapsto p , \end{aligned}$$

which makes TM into a *vector bundle*. This can be seen by constructing the local trivialisations:

$$\begin{aligned} \psi_p : U_p \times \mathbb{R}^m &\to \pi^{-1}(U_p) \\ (q, (v_1, \ldots, v_n)) &\mapsto \left(q, \sum_{i=1}^n v_i \partial_{x^i}\big|_q\right) , \end{aligned}$$

where $(U_p, \phi \equiv \{x^i\})$ is a coordinate neighbourhood of p and $\partial_{x^i} \equiv \partial_i$ are the corresponding coordinate vector fields for TM, defined as derivations by

$$\partial_i\big|_q f = \frac{\partial f \circ \phi}{\partial x^i}(\phi^{-1}(q)) .$$

Moreover, TM inherits a smooth differentiable structure from M via the coordinate charts $(\pi^{-1}(U), (x^i, dx^i))$, where $(U, \{x^i\})$ is a coordinate chart on M, and the coordinates on

[5] All unsubstantiated claims should be considered (possibly rather difficult) exercises!

$\pi^{-1}(U)$ are defined by

$$\left(x^i, dx^j\right) : \left(p, v^i\partial_{x^i}\big|_p\right) \mapsto \left(x_i(p), v^j\right) .$$

This makes TM a smooth $2n$-dimensional differentiable manifold.

We may similarly construct the *cotangent bundle* T^*M from the *cotangent spaces*[6] $T_p^*M := (T_pM)^*$. Analogous procedures as above provide the cotangent bundle with the structures of a smooth vector bundle and a smooth $2n$-dimensional differentiable manifold.

Several constructions on vector spaces extend to vector bundles by defining them fibre-wise and using naturally induced local trivialisations. For example, we constructed T^*M, the *dual bundle* of TM, its fibres being the dual vector spaces of the fibres of TM. Also of importance is the tensor product, $\otimes$, and direct sum, $\oplus$, of two vector bundles, whose fibres are respectively the tensor product and direct sum of the fibres of each bundle in the product resp. sum. The tensor bundle over M is the vector bundle generated by TM and the operations $*$, $\oplus$ and $\otimes$. So its elements are all the possible sums of tensor products of tangent vectors and cotangent vectors.

Vector fields on M are *smooth sections* of TM: smooth maps $X : M \to TM$ such that the vector $X(p)$ is 'attached to M at p', that is, $\pi(X(p)) = p$. The set $\Gamma(TM)$ of smooth sections of TM forms a module over the ring of smooth functions. Sections of the tensor bundle form an algebra (with product $\otimes$) over the ring of smooth functions, called the tensor algebra.

We remark that the *universal property* allows us to identify tensors with corresponding multilinear maps between cartesian products of vector bundles. For example, we can identify an endomorphism $L : TM \to TM$ with a tensor $L \in T^*M \otimes TM$ by identifying TM with TM^{**} and setting

$$L(X, \theta) := \theta(L(X)) .$$

3. The Lie Derivative

Associated with any vector field $X \in \Gamma(TM)$ are its integral curves. These are the solutions $\phi : I \to M$ of the ODEs:

$$\phi' = X \circ \phi .$$

The *flow* of a X is the one parameter family of diffeomorphisms $\phi : M \times I \to M$ generated by the integral curves:

$$\phi_p' = X \circ \phi_p ,$$

where we have denoted $\phi(p, s) = \phi_p(s)$. That is, for each $s \in I$, the diffeomorphism $\phi_s = \phi(\cdot, s)$ sends a point p a distance s along the integral curve of X through p.

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[6]Recall that, given a real linear space V, its *dual space* V^* is the linear space of linear maps from V to $\mathbb{R}$.

The Lie derivative $\mathcal{L}_X Y$ of a vector field Y with respect to a vector field X is the vector field obtained by differentiating Y along the flow of X. That is,

$$\mathcal{L}_X Y\big|_p := \frac{d}{ds}\bigg|_{s=0} \phi_p^* Y = \frac{d}{ds}\bigg|_{s=0} Y \circ \phi_p \,,$$

where *phi* is the flow of X (and we apologise for the subscript gymnastics). In fact, we could replace Y by any section (e.g a tensor) to obtain other Lie derivatives.

It turns out that the Lie derivative of two vector fields is just their commutator.

Difficult Exercise. *Let X and Y be two vector fields. Define their* commutator $[X, Y]$ *by*

$$[X, Y]f = XYf - YXf$$

for any function f.

(a) *Show that $[X, Y]$ is a derivation on $C^\infty(M)$. That is, $[X, Y]$ is a vector field.*
(b) *Show that*

$$\mathcal{L}_X Y = [X, Y]\,.$$

4. Connections and Covariant Differentiation.

Recall that, in Euclidean space, we can differentiate a vector field X in the direction of a vector v, by differentiating X along the line defined by v, that is,

$$D_v X\big|_p := \frac{d}{ds}\bigg|_{s=0} (X(p + sV)) = \lim_{s\to 0} \frac{X(p + sv) - X(p)}{s}\,.$$

We might try to generalise this to a manifold as follows: Let $\gamma : I \to M$ be any curve whose initial tangent is $v = \gamma'(0)$. Then consider

$$D_v X := \frac{d}{ds}\bigg|_{s=0} (X \circ \gamma) = \lim_{s\to 0} \frac{X(\gamma(s)) - X(\gamma(0))}{s}\,.$$

Of course, this doesn't make any sense–$X(\gamma(s))$ and $X(\gamma(0))$ live in different tangent spaces, and hence their difference is undefined.

Note that the Lie derivative is not sufficient here, since $\mathcal{L}_X Y\big|_p$ depends on values of both vector fields in a neighbourhood of p, so that values of Y along a curve, and the value of X at p are not enough.

What we need is a way of identifying, or 'connecting', different tangent spaces along the curve; that is, a family of isomorphisms

$$\tau_s : T_{\gamma(s)}M \to T_{\gamma(0)}M\,.$$

Then we could define

$$\frac{d}{ds}\bigg|_{s=0} (X \circ \gamma) = \lim_{s\to 0} \frac{\tau_s X(\gamma(s)) - X(\gamma(0))}{s}\,.$$

4.1. **Connections.**

A connection on TM is a map

$$\begin{aligned}\nabla : TM \times \Gamma(TM) &\to TM\\ (v, X) &\mapsto \nabla_v X \in T_{\pi(v)}M\end{aligned}$$

that satisfies the following rules

$$\begin{aligned}\mathbb{R}\text{-linearity:}\quad & \nabla_v(aX+Y) = a\nabla_v X + \nabla_v Y\,;\\ \text{Leibniz rule:}\quad & \nabla_v(fX) = (vf)X(p) + f(p)\nabla_v Y\,;\\ \text{Smoothness:}\quad & \nabla_X Y \in \Gamma(TM)\,;\\ C^\infty\text{-linearity:}\quad & \nabla_{fY+Z}X = f\nabla_Y X + \nabla_Z X\,.\end{aligned}$$

for all $a \in \mathbb{R}$, $u, v \in T_pM$, $f \in C^\infty(M)$, $X, Y, Z \in \Gamma(TM)$ and $p \in M$, where $\nabla_X Y(p) := \nabla_{X(p)}Y$. Given a vector (field) X, the map ∇_X is called *covariant differentiation* with respect to X.

Now consider some local coordinates $(U, \{x^i\}_{i=1}^n)$ on a neighborhood U of a point $p \in M$. Then, for each $Y \in \Gamma(TU)$, we may write $Y = Y^i\partial_i$, for some $Y^i \in C^\infty(U)$. Then, for any $v \in T_pM$, we have

$$\begin{aligned}\nabla_v Y = \nabla_v Y^i X_i &= (vY^i)X_i(p) + v^iY^j(p)\nabla_{X_i(p)}X_j\\ &= (vY^i)X_i(p) + v^iY^j(p){\Gamma_{ij}}^k(p)X_k(p)\,.\end{aligned}$$

for some n^3 functions ${\Gamma_{ij}}^k$. It follows that ∇ is completely determined (on U) by these functions. Conversely, we can specify some functions ${\Gamma_{ij}}^k$ on coordinate charts, and glue them together to form a connection on M; there are, in general, many ways to do this. Thus, given a manifold M, there is no canonical connection on TM, and hence no canonical way to specify a 'directional derivative' ∇_v of vector fields.

4.2. **Parallel Transport.**

Let $\gamma : I \to M$ be a smooth curve. We may 'pull the connection back' along γ to define a connection on TI by defining

$$\nabla_{\partial_s}(Y\circ\gamma)\big|_s = \nabla_{\gamma'}Y\big|_{\gamma(s)}$$

for $Y \in \Gamma(TM)$, where s is the curve parameter and ∂_s a basis for TI. We'll denote $\nabla_{\partial_s} = \nabla_s$. If $Y \in \Gamma(\gamma^*TM)$ is a vector field along γ such that $\nabla_s Y = 0$, we say that Y is *parallel* along γ. Conversely, we may consider the differential equation

$$\nabla_{\gamma'}Y = 0\,.$$

Utilising local coordinates $\{x^i\}$, we may expand γ' and Y in terms of the local coordinate basis $\{\partial_i\}_{i=1}^n$ as follows

$$
\begin{aligned}
0 = \nabla_{\gamma'} Y &= (\gamma' Y^k)(X_k \circ \gamma) + {\gamma'}^i (Y^j \circ \gamma)({\Gamma_{ij}}^k \circ \gamma)(\partial_k \circ \gamma) \\
&= \left(\frac{d(Y^k \circ \gamma)}{ds} + \frac{d\gamma^i}{ds}(Y^j \circ \gamma)({\Gamma_{ij}}^k \circ \gamma) \right)(\partial_k \circ \gamma)\,,
\end{aligned}
$$

where $\gamma^k := x^k \circ \gamma$. This is a system of n first order ODEs for the functions $Y^k \circ \gamma : I \to \mathbb{R}$, which, given n initial conditions $Y(\gamma(0)) := v$, we can solve uniquely (on any coordinate patch, and thereby, using a partition of unity, also globally).

This observation allows us to identify nearby tangent spaces: given $p, q \in M$, $v \in T_pM$, and a curve $\gamma : [0,1] \to M$ joining $p = \gamma(0)$ and $q = \gamma(1)$, we define the *parallel translate* of v to T_qM by

$$\tau_{p,q}(v) = V(q)\,,$$

where V is the unique parallel extension of v along γ; that is, the unique solution of

$$
\begin{aligned}
\nabla_{\gamma'} V &= 0 \\
V(p) &= v\,.
\end{aligned}
$$

This map defines an isomorphism between different tangent spaces. However, given different curves, we obtain different isomorphisms. We shall see the deviation of parallel transport along different curves characterises the *curvature* of the connection.

Moreover, given the parallel transport map, we can recover the connection:

$$\nabla_s Y|_{s=0} = \lim_{s \to 0} \frac{\tau_{\gamma(s),p}(Y(s)) - Y(0)}{s}\,.$$

4.3. **Geodesics.**

A curve $\gamma : I \to M$ is said to be *geodesic* if its tangent vector is parallel along itself. That is, if

$$\nabla_{\gamma'} \gamma' = 0\,. \tag{4.1}$$

We can again expand this locally in terms of a coordinate basis $\{\partial_i\}_{i=1}^n$, obtaining

$$0 = \nabla_{\gamma'} \gamma' = \left(\frac{d^2 \gamma^k}{ds^2} + \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}({\Gamma_{ij}}^k \circ \gamma) \right) \partial_k \circ \gamma\,.$$

This is a system of second order ODEs for the components γ^k, which has a unique solution given initial position $\gamma(0) = p$ and velocity $\gamma'(0) = v$.

4.4. **Torsion.**

The Leibniz rule ensures that the map $X, Y \mapsto \nabla_X Y$ is not a tensor, since it is not C^∞-linear in the 'Y' argument. However, it *is* C^∞-linear in the 'X' argument. Now, if we antisymmetrise in X and Y we get

$$\begin{aligned}\nabla_X(fY) - \nabla_{fY}X &= (Xf)Y + f\nabla_X Y - f\nabla_X Y \\ &= f(\nabla_X Y - \nabla_Y X) + [X, fY]\,,\end{aligned}$$

where $[X, Y]$ is the *commutator* of X and Y, which is the vector field defined (as a derivation[7].) by

$$[X, Y]f = X(Yf) - Y(Xf).$$

It follows that the map

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is a tensor. T is called the *torsion tensor* of ∇.

4.5. **Curvature.**

Although the second iterated covariant derivative operator $X, Y, Z \mapsto \nabla_X(\nabla_Y Z)$ is not tensorial either, we can construct a tensor which is second order with respect to ∇ in a similar manner to the construction of T. This tensor is called the *curvature tensor* of ∇. It is defined by

$$X, Y, Z \mapsto R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\,.$$

We leave it as an exercise to prove that this operator is in fact C^∞-linear in X, Y and Z.

In $\mathbb{R}^n$, we expect that second (coordinate) derivatives should compute. We can't expect this to be true in general, and the curvature tensor quantifies the obstruction. In fact, the curvature tensor can also be seen to measure the lack of commution of parallel transport.

4.6. **Covariant Differentiation of Tensors.**

We may covariantly differentiate covector fields $\theta \in \Gamma(T^*M)$ by 'commuting ∇ with contractions':

$$(\nabla_X \theta)(Y) := X\left(\theta(Y)\right) - \theta\left(\nabla_X Y\right)\,.$$

Defining the covariant derivative of a function f to simply be the derivative with respect to the vector field: $\nabla_X f := Xf$, the above definition can be easily remembered as a kind of Leibniz rule:

$$\nabla_X\left(\theta(Y)\right) = (\nabla_X \theta)(Y) + \theta\left(\nabla_X Y\right)\,.$$

[7]It is a simple exercise to show that $[X, Y]$ is a derivation

We can define the covariant derivative of any tensor by generalising this rule. For example, if T is once contra- once co-variant, then

$$(\nabla_X T)(\theta, Y) = X\left(T\left(\theta, Y\right)\right) - T\left(\nabla_X \theta, Y\right) - T\left(\theta, \nabla_X Y\right) .$$

5. Metric Tensors.

A metric tensor g is a smooth assignment of a pseudo-Euclidean structure to each tangent space of M. That is, g is a smooth, symmetric, non-degenerate section of $T^*M \otimes T^*M$. We call the pair (M, g) a *pseudo-Riemannian manifold*. Since g is non-degenerate, it may be used to set up isomorphisms between tensor spaces of the same rank. To see this for tensors of rank one (i.e vectors and their duals, the covectors), we consider the map that takes a vector v to the covector $\widetilde{v} := g(v, \cdot)$. That is, for any other vector u, we have $\widetilde{v}(u) = g(u, v)$. We leave it to the reader to prove that this is an isomorphism of each T_pM with T_p^*M. By taking the inverse isomorphism, we can define a metric on $T^a stM$ by hitting the corresponding vectors with the metric on TM. This generalises to other tensors in the natural way–by expanding each tensor into its homogeneous parts (those which are just tensor products of vectors and covectors) and defining the inner product of two homogeneous tensors of the same type as the product of the inner products of their vector and covector factors. We then complete the picture by distributing over addition. For example,

$$g(u \otimes \theta, v \otimes \alpha + w \otimes \beta) = g(u, v)g(\theta, \alpha) + g(u, w)g(\theta, \beta) .$$

Of course, we can extend this further, to the whole tensor algebra, by distributing further over direct sums.

5.1. The Levi-Civita Connection.

Given a manifold M equipped with both a metric g and a connection ∇, we say that ∇ is compatible with g, or ∇ *is metric compatible*, if parallel translation is an isometry. This holds if and only if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) .$$

Exercise. *Show that the operator*

$$(X, Y, Z) \mapsto Xg(Y, Z) - g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

is a tensor, which we denote

$$\nabla g : (X, Y, Z) \mapsto (\nabla_X g)(Y, Z) := Xg(Y, Z) - g(\nabla_X Y, Z) + g(Y, \nabla_X Z) .$$

Exercise.

(1) *Show that, given a pseudo-Riemannian manifold (M, g), M admits a unique torsion-free ($T = 0$), metric compatible connection ∇, called the* Levi-Civita *connection.*
(2) *Show that, in fact, given any $A \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, and $B \in \Gamma(T^*M \otimes T^*M \otimes TM)$, M admits a unique connection with metric compatibility $\nabla_X g(Y, Z) = A(X, Y, Z)$ and torsion $T(X, Y) = B(X, Y)$ (so long as A and B have the correct symmetries).*

From now on, we shall only consider pseudo-Riemannian manifolds equipped with the Levi-Civita connection.

5.2. **Symmetries of the Curvature Tensor.**

We may identify the curvature tensor with a covariant rank 4 tensor using the metric:

$$R(X,Y,Z,W) := g(R(X,Y)Z,W)\,.$$

Exercise. *Clearly the curvature tensor has the anti-symmetry* $R(X,Y,Z,W) = -R(Y,X,Z,W)$. *Show that it has the following additional symmetries:*

(1) *Since* ∇ *is torsion free:*

$$\underset{X,Y,Z}{\Xi}\, R(X,Y)Z := R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0\,,$$

and

$$\underset{X,Y,Z}{\Xi}\, \nabla_X R(Y,Z)W = 0\,,$$

where we recall that ∇R *is the tensor defined by*

$$\nabla_X R(Y,Z)W := \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)(\nabla_X W)\,.$$

(2) *Since* ∇ *is metric compatible,*

$$R(X,Y,Z,W) = \,-\,R(X,Y,W,Z)\,.$$

(3) *Since* ∇ *is both torsion free and metric compatible,*

$$R(X,Y,Z,W) = R(Z,W,X,Y)\,.$$

(4) *Deduce that* R *has only one non-trivial trace (up to a minus sign), the* Ricci tensor*:*

$$\mathrm{Ric}(X,Y) := \mathrm{Tr}\,\left(Z \mapsto R(Z,X)Y\right) = \theta^i(R(X_i,X)Y)\,,$$

where $\{X_i\}_{i=1}^n$ *and* $\{\theta^i\}_{i=1}^n$ *are a pair of dual bases, that is,* $\theta^i(X_j) = \delta^i{}_j$. *The trace of the Ricci tensor is called the* scalar curvature *of* ∇*:*

$$\mathcal{R} := \mathrm{Tr}\,\left(X \to \mathrm{Ric}(X)\right) = \mathrm{Ric}(X_i,\theta^i) = g^{ij}\mathrm{Ric}(X_i,X_j)\,,$$

where we have identified the twice covariant Ric *with the corresponding endomorphism of* TM *using the metric.*

(5) *Show, using the second Bianchi identity, that*

$$\mathrm{div}\,\mathrm{Ric} = \,-\,\frac{1}{2}d\mathcal{R}\,,$$

where the divergence operator is defined by the rule: 'differentiate and contract', for example,

$$\mathrm{div}\,\mathrm{Ric} := g^{ij}(\nabla_i\mathrm{Ric})(X_j)\,,$$

and the differential (a.k.a exterior derivative) d *takes functions to one forms via the rule* $df(X) := Xf$.

References

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