Theorem 1. If $G$ is a compact semisimple Lie group and $T$ is a maximal torus, every $g \in G$ is conjugate to some $t \in T$. Also if $t, t' \in T$ are conjugate in $G$ then

$$t' = g t g^{-1}$$

for some $g \in N(T)$.

Since $G$ acts on itself by conjugation, it acts on $g$:

$$\text{Ad}: G \to \text{Aut}(g)$$

Every $x \in g$ is "conjugate" to some $t \in T$:

$$x = \text{Ad}(g)t$$

for some $g \in G$. 
Also if \( t \in T \) are "conjugate" in \( G \), i.e. \( t' = \text{Ad}(g)t \) for some \( g \in G \), then we can actually choose \( g \in N(T) \).

Recall \( N(T) \) acts on \( T \) by conjugation and thus on \( t \):

\[
\text{Ad}_{N(T)} : N(T) \rightarrow \text{Aut}(T)
\]

Since \( T \subseteq N(T) \) acts trivially on itself by conjugation, the Weyl group

\[
W = \frac{N(T)}{T}
\]

acts on \( T \) by conjugation and on \( t \) by \( \text{Ad} \).

So by our thm:

\[\{ \text{conjugacy classes in } G \} \cong \{ \text{W orbits in } T \}\]

\[\{ \text{adjoint orbits in } G^3 \} \cong \{ \text{W orbits in } T^3 \}\]
Examples

$A_1 = \text{SU}(2)$. Here $G = \text{SU}(2)$

\[
T = \left\{ \left( \begin{array}{cc}
e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right) : \theta \in \mathbb{R} \right\}
\]

$\text{SU}(2) \cong \{ \text{unit quaternions} \} \cong S^3 \subset \mathbb{H}$

$\text{SU}(2)$

Each unit imaginary quaternion gives such a circle; $\mathbb{RP}^2$ is the space of maximal tori.

$\text{Spin}(3) \cong \text{SU}(2) \frac{2-1}{2-1} \cong S^0(3)$
$Z(\text{SO}(2)) = \{ \pm 1 \}$, \quad $\text{SO}(2)/\{ \pm 1 \} \cong \text{SO}(3)$

Two elements of $\text{SO}(3)$ are conjugate iff they're rotations about some axis by the same (unsigned) angle.

Conjugacy classes in $\text{SU}(2)$ are (typically) 2-spheres like this:
Also: \( \{1\} \cong \{ -1 \} \) are conjugacy classes.

So

\( \{ \text{conjugacy classes in } SU(2) \} \cong [0, \pi] \)

Another way to see it:

\[
\begin{array}{cccc}
-2 & -1 & 0 & 1 & 2 \\
\text{T} & \cong & S^1 \\
\text{t} & \cong & \mathbb{R} \\
\text{U1} & \cong & L \\
\frac{t}{L} & \cong & T
\end{array}
\]

For \( A_n \), the Weyl group is \( S_{n+1} \).

For \( A_1 \), it's \( S_2 \cong \mathbb{Z}_2 \), acting as reflection on \( t \).
So:

3 adjoint orbits in \( g^3 \cong \{ W \text{ orbits in } t^3 \cong [0, \infty) \}

3 conjugacy classes in \( G^3 \cong \{ W \text{ orbits in } T^3 \cong \{ W \text{ orbits in } T/L \cong [0, \pi] \}

\[
\begin{array}{cccccc}
\text{t} & -2 & -1 & 0 & 1 & 2 \\
\text{W} & 0 & 1 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{exp}(2\pi i) & -1 & 1 & T \\
T/W & -1 & 1 & T/W \\
\end{array}
\]

D_2, A_0(4), Spin(4)

\[\theta, t\]

\[W \cong \mathbb{Z}_2 \times \mathbb{Z}_2\]
\[ W \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \]

**What's**

\[ \{ \text{adjoint orbits in } G^3 \} \cong \{ \text{W orbits in } T^3 \} = \]

\[ = \{ (\theta, \delta) \in \mathbb{R}^2 : 0 \leq \theta, \delta \leq |\theta| \} \]

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**A Weyl Chamber** is a fundamental region for the \( W \) action on \( T \) whose "walls" are the "mirrors" for the reflections generating \( W \).

**What's**

\[ \{ \text{conj. classes in } G^3 \} = \{ \text{W orbits in } T^3 \} = \]